Stability Gap between Off- and On-Firing States in a Coupled Ginzburg-Landau Oscillator Neural Network

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An oscillator neural network based on the Ginzburg-Landau equation is proposed in order to investigate associative memory. The embedded patterns include both off- and on-firing states with phase. Neurons encode not only the phase but also the state of the cell. The system has a Lyapunov function with wells corresponding to off- and on-firing states. These two states are characterized by mutual stabilities which are generally different. Each neuron falls into one of the wells, as determined by its dynamics.

The storage capacity and the order parameter overlap are studied using numerical simulations and an ordinary self-consistent signal-to-noise analysis. We show that these properties depend crucially on both the potential parameter and the mean activity levels of the patterns. In the case of the sparse coding limit, the storage capacity diverges as in the case of the binary model.

§1. Introduction

There are many reports on biological synchronization phenomena, e.g., in the circadian rhythm of animals, heart tissue, plasmodium and light emission from fireflies. There are also several experimental observations on the phase synchronization of oscillatory responses in the brain. It is well known that when neurons respond to external stimuli, the firing rate increases. When both the firing rate and its timing (phase) are considered, an oscillatory neuronal firing and its synchronization between spatially separated cells in the visual cortex were observed. Based on those observations, it was proposed that such phase synchronization may serve as a mechanism for the extraction and representation of features of a pattern.

As a second important example, Vaadia et al. observed the neuronal firing rate and its timing for spatially separated cells in response to stimuli using trained monkeys. The results suggest that the sole observation of the firing rate is not sufficient for understanding the mechanism of the neural coding. That is, the correlation between distant neuronal firings and/or their temporal structures also include important information. From these recent experiments, we can see that not only the firing rate but also its relative timing play crucial roles in the information processing of the brain.

Since the pioneering work on coupled oscillator models, there have been many theoretical works on the phase synchronization of systems with many oscillators. As their extension to neural networks, the uniform coupling term is replaced by a Hebbian learning rule in the synaptic connection. This aspect has been extensively investigated in the contexts of statistical physics and applied mathematics.
From the engineering point of view, the most promising application of such models is pattern recognition using associative memory. In conventional neurocomputers with \( n \) neurons, the system would have \( O(n^2) \) connections. For large \( n \), it is very difficult to treat. However, we can reduce the number of connections significantly using dynamic connectivity in the oscillatory neurocomputer. For this purpose, oscillatory neurocomputer models based on the rhythmic behaviour of the brain have been investigated.\(^{14,15}\)

We would like to understand the properties of oscillatory neuron assemblies. Let us consider the situation where there are many interacting neurons in the on-firing state. We assume that the frequencies of the neurons are different. If these frequencies vary greatly, they do not communicate and fire independently. However, when they are almost the same, each neuron is sensitive to the phases of the other neurons, and they interact strongly. In this case, all neurons oscillate with the same frequency and fixed phase. Such a network can memorize information through temporal coding.\(^{16}\)

The above cited theoretical works on phase coupled oscillator models of associative memory assume a priori that all neurons are always in the on-firing state, and the properties of these models with this assumption were studied. However, the assumption that all neurons are firing all the time is not realistic. In a real neuronal system, some neurons are in the on-firing state, while others are in the off-firing state, depending on the particular situation. In this way, a neural network balances its mean firing rate with the mean activity level in the embedded patterns. According to results of biological studies, this is due to the feedback mechanism between outputs from the whole associative neural network and the inhibitory neuron group. There are several experimental observations on such a feedback control system.\(^{17,18}\)

It is well known that the cell state has a nonlinear dependence on the membrane potential. A neuron has a threshold for firing: It fires with an action potential only if the level of external stimuli exceed this threshold. In the present study, we do not introduce such a firing threshold. Instead, we use the relative stability. In order to realize both off- and on-firing states, we propose a system that has a Lyapunov function with wells corresponding to each state. The shape of the Lyapunov function changes with the value of the potential parameter. This creates the stability gap between the states of a cell.

There are many studies on Ginzburg-Landau equations with a mean field term.\(^{19\text{-}22}\) These equations are mathematically tractable, as each oscillator is coupled symmetrically to all others. Contrary to it, there are few studies of neural oscillator models of associative memory. Aoyagi proposed a Stuart-Landau equation type model with a pseudo-inverse connection in the Hebbian term and showed numerically that it is a good candidate for associative memory.\(^{23}\) Uchiyama and Fujisaka proposed a new analytical method for an oscillator neural network without a Lyapunov function.\(^{24}\) If this system has a Lyapunov function, it reaches one unique state, where the free energy realizes a minimum value. However, in the case without a Lyapunov function, other criteria must be used in order to single out the relevant solution. They proposed a new criterion based on one-dimensional map theory and applied it to a coupled complex Ginzburg-Landau equation. They con-
cluded that the dynamical theory should be taken into consideration to understand the whole phase diagram.

Recently, the properties of oscillator neural networks which encode both the state of the cell and its phase have been reported.\textsuperscript{25} The authors of that paper investigated a typical oscillator neural network model with a threshold parameter which realizes off- and on-firing states. However, there seems to be no study on a model that considers the relative stability among the states of a cell rather than the firing threshold. From the above described point of view, we propose an associative memory model based on the Ginzburg-Landau equation. The embedded patterns are assumed to include off- and on-firing states with phase.

The purpose of the present research is to investigate how the network retrieves embedded patterns, and we discuss the dependence of the storage capacity on the potential parameter and the activity level in the patterns.

\section{Model equation}

We describe the state of the \( i \)-th oscillatory neuron by \( W_i \) (\( i = 1, 2, \ldots, N \)), where \( N \) is the total number of neurons. We assume that each \( W_i \) is a complex variable. Its amplitude and phase correspond to the firing strength and phase of oscillation, respectively. Then, we can denote \( W_i \) as \( W_i = R_i \exp(i\Phi_i) \), where \( R_i \geq 0 \) and \( 0 \leq \Phi_i < 2\pi \). We define \( R_i = 0 \) and \( R_i = 1 \) to be the off- and on-firing states, respectively.

Now, we consider the situation where each neuron evolves with nonlinear dynamics and interactions. We propose the following coupled oscillators model based on the Ginzburg-Landau equation:\textsuperscript{23, 26-28}

\[
\frac{dW_i}{dt} = v(W_i) + \left( \sum_{j=1}^{N} C_{ij} W_j - W_i \right),
\]

\[
v(W_i) = i\omega_i W_i + kW_i(1 - |W_i|^2)(-a^2 + |W_i|^2),
\]

where \( \omega_i \), \( k \) and \( a \) are the natural frequency, the positive coefficient of the nonlinear term and the potential parameter, respectively. In the present study, we assume that every neural oscillator has the same frequency \( \Omega \). Then, we can eliminate it in the rotating framework. The variable \( a \) is assumed to be in the interval \( 0 \leq a \leq 1 \).

The synaptic connection \( C_{ij} \) is given by the embedded pattern \( \xi_i^k \) according to

\[
C_{ij} = \frac{1}{qN} \sum_{k=1}^{p} \xi_i^k \tilde{\xi}_j^k,
\]

where \( p \) is the total number of patterns and \( q \) is the mean activity level in the patterns, satisfying \( 0 \leq q \leq 1 \). Here, \( C_{ij} \) corresponds to the generalized Hebbian rule, and we note that it is Hermitian: \( C_{ij} = \bar{C}_{ji} \).

We define the embedded pattern \( \xi_i^\mu \) as

\[
\xi_i^\mu = A_i^\mu e^{i\theta_i^\mu},
\]
where $\theta^\mu_i$ is a random variable, distributes uniformly in the interval $0 \leq \theta^\mu_i < 2\pi$. $A^\mu_i$ is also random variable taking either the value 0 or 1, corresponding to the off- and on-firing state, respectively. Thus, the embedded patterns include both the state of the cell and its phase.

We note that $A^\mu_i$ satisfies the probability distribution

$$P(A^\mu_i) = q\delta(A^\mu_i - 1) + (1 - q)\delta(A^\mu_i).$$

(2.4)

Then, we define the overlap $m^\mu$ between the state of network and embedded pattern $\mu$ by

$$m^\mu = \frac{1}{qN} \sum_{j=1}^{N} \tilde{G}^\mu_j W_j.$$

(2.5)

If each neuron evolves according to (2.1) starting from a noisy pattern, the network reaches an equilibrium state after a sufficient time. In order to retrieve stably an embedded pattern, such a state should be the attractor of the dynamics. In this model, the system has a Lyapunov function,

$$L = \sum_{i=1}^{N} V(W_i, \tilde{W}_i)
- \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} (C_{ij} \tilde{W}_i W_j + \tilde{C}_{ij} W_i \tilde{W}_j)
n+ \sum_{i=1}^{N} |W_i|^2,$$

(2.6)

where

$$V(W_i, \tilde{W}_i) = k \left( \frac{1}{3} |W_i|^6 - \frac{1}{2} (a^2 + 1) |W_i|^4 + a^2 |W_i|^2 \right),$$

(2.7)

satisfying $v = -\partial V(W_i, \tilde{W}_i)/\partial W_i$. Using the relation $dW_i/dt = -\partial L/\partial \tilde{W}_i$ and (2.1), we can show that $L$ is a monotonically decreasing function of time, $dL/dt \leq 0$. This suggests that if the network converges to an embedded pattern, the pattern is locally stable.

We can see that the potential $V(W_i, \tilde{W}_i)$ is an even function of $|W_i|$ and has two wells in the interval $|W_i| \geq 0$. Although the shape of the potential depends on the potential parameter $a$, the positions of the local minima remain at $|W_i| = 0, 1$, corresponding to the off- and on-firing states, respectively, as shown in Figs. 1(a)–(c). This suggests that both states are attractors of the dynamics so that every neuron asymptotically reaches either of them in the sense that $\lim_{t \to \infty} R_i = 0$ or 1. The difference in energy between these states is a measure of their relative stability. For larger $a$, the off-firing state is more stable than the on-firing state, and vice versa. When $a = 1/\sqrt{3}$, their stabilities are equal.
Fig. 1. Potential $V$ as a function of $|W|$ for typical values of $a$. Here $k = 5.0$. (a) $a = 1/\sqrt{3} - 0.15$. (b) $a = 1/\sqrt{3}$. (c) $a = 1/\sqrt{3} + 0.15$.

§3. Simulations and theoretical results

In order to investigate the statistical properties of the system for a large population, we calculate the macroscopic order parameters in the equilibrium state. We study how the properties of the network, as a system with associative memory, are influenced by the potential shape and the mean activity level in the patterns.

Let us consider the effect of the parameter $k$ on the system. If $k$ is very small, the effect of the nonlinear term $v$ is small. In this case, each cell converges to neither $R_i = 0$ nor $1$ in the equilibrium state, so that the system cannot retrieve embedded patterns. Contrary to it, when $k$ is very large, the potential barriers are very high so that there occurs no transition between off- and on-firing states. In this situation with dense coded patterns, we checked that the system behaves similarly to the phase coupled model.\(^{12}\) In our study, in order to balance the nonlinear term with the coupling term in (2.1), we choose $k = 5$.

We compare numerical simulations with theoretical results for typical values of $a$ and $q$. The self-consistent signal-to-noise analysis (SCSNA) enables us to study theoretically the properties in the equilibrium state.\(^{29}\) This analysis consists of a kind of systematic renormalization procedure, for which the Hebbian term in (2.1) is decomposed into signal, Gaussian noise and output proportional parts. Order parameter equations are derived and solved self-consistently.

When we treat the macroscopic properties theoretically, we consider the situation where $p, N \to \infty$ with fixed loading rate $\alpha (= p/N)$. We assume that $\mu = 1$ is the only condensed pattern, the others are all uncondensed patterns; that is, $m^1 = O(1)$ and $m^\mu \sim O(1/\sqrt{N})$ (for $\mu > 1$).

We separate $m^\mu$ and $\xi^\mu$ into real and imaginary parts as

\[ m^\mu = m_c^\mu + im_s^\mu \]  \hspace{1cm} (3.1)  \\
and  \\
\[ \xi^\mu = \xi_c^\mu + i\xi_s^\mu. \]  \hspace{1cm} (3.2)
Hereafter, we omit the superscript 1 of $m_c$, $m_s$, $\xi_c$ and $\xi_s$ for simplicity.

Following the original SCSNA and its extension to complex variables, we obtain a set of formulas for the macroscopic order parameters which describes the equilibrium state. The outline of its derivation is given in the Appendix. Here we give only the final formulas and their numerical solutions.

Let $R(x_1, x_2)$ and $\Phi(x_1, x_2)$ be the amplitude and the phase of $W$ in the equilibrium state, respectively. They are determined by the solution of

\[
(m_c \xi_s + m_s \xi_c + x_2) \cos(\Phi) - (m_c \xi_c - m_s \xi_s + x_1) \sin(\Phi) + \alpha RS_i / \Lambda = 0 \quad (3.3)
\]

and

\[
kR(1 - R^2)(-a^2 + R^2) + (m_c \xi_c - m_s \xi_s + x_1) \cos(\Phi)
+ (m_c \xi_s + m_s \xi_c + x_2) \sin(\Phi)
- R + \alpha R(1 - S_r) / \Lambda = 0,
\quad (3.4)
\]

where

\[
\Lambda = (1 - S_r)^2 + S_i^2,
\quad (3.5)
\]

\[
S_r = \frac{\sigma^2}{2} \langle \langle R(x_1 \cos(\Phi) + x_2 \sin(\Phi)) \rangle \rangle_\xi
\quad (3.6)
\]

and

\[
S_i = \frac{\sigma^2}{2} \langle \langle R(x_1 \sin(\Phi) - x_2 \cos(\Phi)) \rangle \rangle_\xi.
\quad (3.7)
\]

Here, the angle brackets $\langle \langle \cdots \rangle \rangle_\xi$ denote an average over $\xi$ and Gaussian integration over $x_1$ and $x_2$:

\[
\langle \langle \cdots \rangle \rangle = \frac{\sigma^2}{2\pi} \int \int e^{-\frac{1}{2}\sigma^2(x_1^2 + x_2^2)} \cdots \, dx_1 dx_2.
\quad (3.8)
\]

$s^{-2}$, $m_c$ and $m_s$ are determined self-consistently by

\[
\sigma^{-2} = \frac{\alpha}{2\Lambda} \langle \langle R^2 \rangle \rangle_\xi,
\quad (3.9)
\]

\[
m_c = \frac{1}{q} \langle \langle R(\xi_c \cos(\Phi) + \xi_s \sin(\Phi)) \rangle \rangle_\xi,
\quad (3.10)
\]

and

\[
m_s = \frac{1}{q} \langle \langle R(\xi_c \sin(\Phi) - \xi_s \cos(\Phi)) \rangle \rangle_\xi.
\quad (3.11)
\]

The above set of equations are solved self-consistently using the Newton method. As a first procedure, we must find the relevant solution of (3.3) and (3.4). A brief explanation on it is given in the Appendix.

Using the above formulas, we study the dependence of the overlap on the loading rate for typical $a$ and $q$. For $a = 1/\sqrt{3} - 0.15$, theoretical results are consistent with numerical simulations, as shown in Figs. 2(a)-(i) and (a)-(ii). For $a = 1/\sqrt{3}$, there is some discrepancy between simulations and theoretical results, and the storage
Fig. 2. Dependence of the overlap \(|m|\) on the loading rate \(\alpha\). The solid curve and marks represent the theoretical results obtained from SCSNA and numerical simulations, respectively. In the simulations, we used \(N = 1000\) with 5 trials, except in the case of (c), where we used \(N = 1000\) with 10 trials. 

(a)-(i) \(a = 1/\sqrt{3} - 0.15\) and \(q = 1.0\). 
(b)-(i) \(a = 1/\sqrt{3}\) and \(q = 1.0\). 
(c)-(i) \(a = 1/\sqrt{3} + 0.15\) and \(q = 1.0\).

(a)-(ii) \(a = 1/\sqrt{3} - 0.15\) and \(q = 0.8\). 
(b)-(ii) \(a = 1/\sqrt{3}\) and \(q = 0.6\). 
(c)-(ii) \(a = 1/\sqrt{3} + 0.15\) and \(q = 0.2\).
Fig. 3. Dependence of the storage capacity $\alpha_c$ on the mean activity level $q$ for typical values of $a$.

(a) $a = 1/\sqrt{3}$. (b) $a = 1/\sqrt{3} - 0.15$. (c) $a = 1/\sqrt{3} + 0.15$.

capacity $\alpha_c$ obtained from simulations is larger, as shown in Figs. 2(b)-(i) and (b)-(ii). When $a = 1/\sqrt{3} + 0.15$, the off-firing state is more stable than the on-firing one so that it is difficult to retrieve embedded patterns for larger $q$. However, for smaller $q$, it is easier to retrieve patterns, as shown in Figs. 2(c)-(i) and (c)-(ii). Here, the on-firing state is locally stable, but the off-firing state is globally more stable. There are many neurons that are trapped in the on-firing state. This causes wide distribution of simulation data. When $q = 1$, we see that $\alpha_c$ is larger for smaller $a$.

Each neuron reaches either the off- or on-firing state according to the dynamics after a sufficient time. There exists a potential barrier between these two states. If the potential barrier is very high, that is, if one state is much more stable than the other, the more stable state is realized for every neuron. Contrary to it, if the potential barrier is not so high, each neuron in one of the states can make a transition to the other state. However, this can only happen through interactions with the other neurons. Thus, when neurons are easily trapped in spurious states, it is difficult to determine $\alpha_c$ by numerical simulations, even if the system has a Lyapunov function.

The dependence of the storage capacity on the potential parameter and the activity level in the patterns is shown in Fig. 3. It is well known that the storage capacity of the binary information model diverges as $-1/q \ln(q)$ in the limit $q \to 0$. For the case of sparse coding, noise is reduced and spurious states are suppressed so that the storage capacity increases. For our model, Fig. 3 suggests that $\alpha_c$ also diverges for sufficiently smaller $q$. When $a = 1/\sqrt{3}$, there is no stability gap between the two states of a cell, and $\alpha_c$ increases monotonically, as shown in Fig. 3(a). For smaller $a$, the on-firing state is more stable, $\alpha_c$ is a decreasing function of $q$ for smaller $q$. However, it diverges in the limit $q \to 0$, as shown in Fig. 3(b). For larger $a$, the off-firing state is more stable, and it is difficult for every neuron to fire so that $\alpha_c$ remains nearly zero for large $q$. However, when $q$ is smaller, with sufficiently reduced noise, it increases sharply, as shown in Fig. 3(c).

As we have shown above, our model is a candidate for associative memory. The properties of the memory retrieval, such as the overlap and the storage capacity, depend crucially on the potential parameter and the activity level in the patterns.
§4. Conclusion and remarks

We investigated a coupled Ginzburg-Landau oscillator neural network. The system has a Lyapunov function with wells. Neurons are either in off- or on-firing states in the equilibrium. Although both states are attractors of the dynamics, there is a relative stability gap depending on the potential parameter $a$.

If the off-firing state is more stable, when $a$ is larger, the network cannot retrieve densely coded patterns. Contrary to it, if the on-firing state is more stable, when $a$ is smaller, the network can retrieve both sparsely and densely coded patterns. The reason for the discrepancy between numerical simulations and theoretical results is that neurons are trapped in less stable attractors.

The storage capacity $\alpha_c$ has a strong dependence on $q$ and $a$, and it increases for sufficiently small $q$. In the limit $q \rightarrow 0$, we numerically checked that it does not depend on $a$ and diverges as in the binary information model. This suggests that the storage capacity in the sparse coding limit does not depend on the potential structure but on the connectivities among neurons. Aoyagi et al. proposed a simply extended model of the oscillator neural network in which both the state of cell and the relative timing of fire are encoded. They studied the influence of the firing threshold and the mean activity level on dynamical properties of the model. It was concluded that $\alpha_c$ diverges likely to the standard binary model in the sparse coding limit. Moreover, it was found that the basin of attraction remains sufficiently large even near $\alpha_c$. In our model, the system has two attractors with respect to the state of the cell so that there can occur a transition between the off- and on-firing states. The energy barrier between them causes spurious states and the macroscopic properties become complex owing to them. However, in the limit $q \rightarrow 0$, such a potential barrier does not influence the storage capacity.

There are many open problems regarding to our model. We have assumed that each neuron has the same frequency so that we eliminated it in a rotating framework. The effect of a randomly distributed natural frequency on the associative memory will be reported elsewhere. Secondly, if our model is extended to a coupled “complex” Ginzburg-Landau equation, the system does not have a Lyapunov function no longer. The influence of complex terms on the macroscopic parameters should be studied. From our preliminary research, the overlap using ordinary SCSNA does not agree well with numerical simulations. When we treat the system without a Lyapunov function, new methods are needed in order to obtain the relevant solution.

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Appendix

We briefly outline the derivation of the macroscopic order parameter equations which describe the equilibrium state. The equilibrium states of the oscillator neural network are given by \( dW_i/dt = 0 \) \((i = 1, 2, \cdots, N)\) in (2.1). We denote each such state as \( x_i = W_i \) \((t = \infty)\).

It is assumed that \( \mu = 1 \) is the only condensed pattern, and that the other patterns are all uncondensed; that is, \( m^1 = O(1) \) and \( m^\mu \sim O(1/\sqrt{N}) \) \((\mu > 1)\). We denote the local field of the \( i \)-th oscillator such as \( h_i \). The concept of the SCSNA is that the Hebbian term is decomposed into three components. The local field is composed of signal and noise parts:

\[
h_i = \sum_j C_{ij} W_j = \sum_\mu \xi_1^\mu m^\mu = \xi_1^1 m^1 + \sum_{\mu \geq 2} \xi_1^\mu m^\mu. \tag{A.1}
\]

Further, we decompose the second noise term into a pure noise \( z_i \) and a systematic part, the so-called Onsager part \( \gamma x_i \). Then, we have

\[
h_i = \xi_1^1 m^1 + z_i + \gamma x_i. \tag{A.2}
\]

We regard \( x_i \) to be determined by the relation \( x_i = F(h_i, h_i^*) \), where \( F \) is an unknown function and \( h_i^* \) means the complex conjugate of \( h_i \). We assume that this can be solved formally as \( x_i = \tilde{F}(\bar{h}_i, \bar{h}_i^*) \), where \( \bar{h}_i \) is given by

\[
\bar{h}_i = \xi_1^1 m^1 + z_i = \xi_1^1 m^1 + z_{i,\mu} + \xi_1^\mu m^\mu \equiv \bar{h}_i^0 + \xi_1^\mu m^\mu. \tag{A.3}
\]

We note that \( \bar{h}_i \) does not contain the output proportional part. In the pure noise term, there is a small component which is proportional to the \( \mu \)-th signal part so that we separated it in (A.3).

Substituting \( x_i = \tilde{F}(\bar{h}_i, \bar{h}_i^*) \) into \( m^\mu \) defined by (2.5), we expand up to first order in \( m^\mu \) as

\[
m^\mu = \frac{1}{qN} \sum_j \xi_j^{\mu*} x_j
\]

\[
\simeq \frac{1}{qN} \sum_j \xi_j^{\mu*} \tilde{F}(\bar{h}_j^0, \bar{h}_j^{0*})
\]

\[
+ \frac{1}{qN} \sum_j \xi_j^{\mu*} \tilde{F}_1(h_j^0, \bar{h}_j^{0*}) \xi_j^\mu m^\mu + \frac{1}{qN} \sum_j \xi_j^{\mu*} \tilde{F}_2(\bar{h}_j^0, \bar{h}_j^{0*}) \xi_j^\mu m^\mu
\]

\[
= \frac{1}{qN} \sum_j \xi_j^{\mu*} \tilde{F}(\bar{h}_j^0, \bar{h}_j^{0*})
\]

\[
+ \frac{1}{qN} \sum_j |\xi_j^\mu|^2 \tilde{F}_1(h_j^0, \bar{h}_j^{0*}) m^\mu + \frac{1}{qN} \sum_j \tilde{F}_2(\bar{h}_j^0, \bar{h}_j^{0*}) \xi_j^{\mu*2} m^\mu
\]

\[
\equiv \frac{1}{qN} \sum_j \xi_j^{\mu*} \tilde{F}(\bar{h}_j^0, \bar{h}_j^{0*}) + Sm^\mu, \tag{A.4}
\]
where $\tilde{F}_1 = \partial \tilde{F} / \partial \tilde{h}^0_j$, $\tilde{F}_2 = \partial \tilde{F} / \partial \tilde{h}^{0*}_j$ and
\[ S = \frac{1}{qN} \sum_j |\xi^\mu_j|^2 \tilde{F}_1 (\tilde{h}^0_j, \tilde{h}^{0*}_j). \]  
(A-5)

Here, the third term in (A.4) is zero due to the symmetry of the embedded patterns. Thus, we obtain
\[ m^\mu = \frac{1}{(1 - S)} \frac{1}{qN} \sum_j \xi^\mu_j \tilde{F}_1 (\tilde{h}^0_j, \tilde{h}^{0*}_j). \]  
(A-6)

Using (A-6), we decompose the noise term as
\[ \sum_{\nu \neq 1, \mu} \xi^\nu_i m^\nu = \sum_{\nu \neq 1, \mu} \xi^\nu_i \frac{1}{(1 - S)} \frac{1}{qN} \sum_j \xi^\nu_j \tilde{F}_1 (\tilde{h}^0_j, \tilde{h}^{0*}_j) \]
\[ = \sum_{\nu \neq 1, \mu} \xi^\nu_i \frac{1}{(1 - S)} \frac{1}{qN} \sum_j \xi^\nu_j \tilde{F}_1 (\tilde{h}^0_j, \tilde{h}^{0*}_j) \]
\[ + \sum_{\nu \neq 1, \mu} \xi^\nu_i \frac{1}{(1 - S)} \frac{1}{qN} \sum_{j \neq i} \xi^\nu_j \tilde{F}_1 (\tilde{h}^0_j, \tilde{h}^{0*}_j) \]
\[ \equiv \frac{\alpha}{(1 - S)} x_i + z_{i, \mu}. \]  
(A-7)

From the above, we see that $\gamma = \frac{\alpha}{(1 - S)}$. For the pure noise term $z_{i, \mu}$, we can easily check that $\langle z_{i, \mu} \rangle = \langle z^2_{i, \mu} \rangle = 0$ and the variance is
\[ \langle |z_{i, \mu}|^2 \rangle = \frac{1}{|1 - S|^2} \frac{1}{(qN)^2} \sum_{\nu \neq 1, \mu} |\xi^\nu_i|^2 \sum_{j \neq i} |\xi^\nu_j|^2 |\tilde{F}_1 (\tilde{h}^0_j, \tilde{h}^{0*}_j)|^2 \]
\[ = \frac{\alpha}{|1 - S|^2} \frac{1}{N} \sum_{j \neq i} |\tilde{F}_1 (\tilde{h}^0_j, \tilde{h}^{0*}_j)|^2, \]  
(A-8)

where $\langle \rangle$ denotes the average over the random patterns $\xi^\nu_i$ ($i = 1, 2, \ldots, N; \nu \geq 2$). Thus, we can see that the noise obeys a Gaussian distribution, and its mean and variance are obtained. In the limit $N \to \infty$, we expect that the statistical properties of $z_{i, \mu}$ are independent of the site and obey an identical Gaussian distribution.

We would like to find $\tilde{F}_1 (\tilde{h}^0_j, \tilde{h}^{0*}_j)$ in the form $R_j \exp(i \Phi_j)$ and substitute it into the above equations. If we assume the self-averaging property, the site average $1/N \sum_j$ is replaced by the average over the random variable $z = x_1 + i x_2$ and random patterns. Here, $z$ is a single noise obeying a Gaussian distribution. Then we finally obtain Eqs. (3-3)–(3-11).

Now, we briefly describe manipulations and method to solve (3-3)–(3-11). In order to reduce the number of macroscopic order parameters in them, we choose a gauge such that $m_s = 0$. Then, we note that the amplitude of overlap satisfies $|m| = m_c$. Secondly, noting that the system is rotationary symmetric, we rotate it so that $\xi_s = 0$ and $\xi_c = 0$ or 1. Then, we obtain more simple forms:
\[ x_2 \cos(\Phi) - (m_c \xi_c + x_1) \sin(\Phi) + \alpha R S_i / \Lambda = 0 \]  
(A-9)
and
\[
kR(1 - R^2)(-a^2 + R^2) + (m_c\xi_c + x_1)\cos(\Phi) + x_2\sin(\Phi) - R + \alpha R(1 - S_r)/\Lambda = 0, \tag{A.10}
\]
where
\[
\Lambda = (1 - S_r)^2 + S_r^2, \tag{A.11}
\]
\[
S_r = \sigma^2/2 \langle R(x_1\cos(\Phi) + x_2\sin(\Phi)) \rangle, \tag{A.12}
\]
and
\[
S_i = \sigma^2/2 \langle R(x_1\sin(\Phi) - x_2\cos(\Phi)) \rangle. \tag{A.13}
\]

We must look for a relevant solution of (A.9) and (A.10). By squaring (A.9), we obtain \(\cos(\Phi)\) and \(\sin(\Phi)\) as functions of \(R\):
\[
\cos(\Phi) = \frac{-F_1A_2 + A_1\sqrt{A_1^2 + A_2^2 - F_1^2}}{A_1^2 + A_2^2} \tag{A.14}
\]
and
\[
\sin(\Phi) = \frac{F_1A_1 + A_2\sqrt{A_1^2 + A_2^2 - F_1^2}}{A_1^2 + A_2^2}, \tag{A.15}
\]
where \(A_1 = m_c\xi_c + x_1, A_2 = x_2\) and \(F_1 = \alpha RS_i/\Lambda\).

We substitute the above formulas into (A.10). The solutions are determined by the intersections of \(f(R) = g(R)\) in the range \(R \geq 0\), where
\[
f(R) = kR(1 - R^2)(-a^2 + R^2) \tag{A.16}
\]
and
\[
g(R) = -A_1\cos(\Phi) - A_2\sin(\Phi) + R - \alpha R(1 - S_r)/\Lambda. \tag{A.17}
\]

From the graphical representation, we can see that maximum three solutions can be obtained, as shown in Fig. 4. When there are three solutions, one is unstable and the two others are stable. We must single out the relevant solution by applying the Maxwell rule.\(^{30,35}\) It determines the condition for free energy minimum in a system with a Lyapunov function. Using the relevant solution \(R\), we obtain \(\Phi\) from (A.14) and (A.15).

![Fig. 4. Relevant solution among points satisfying \(f(R) = g(R)\). The filled and open circles represent unstable and stable solutions, respectively. According to the Maxwell rule, the stabler solution with larger enclosed area, so that the right open circle is the relevant solution.](http://ptp.oxfordjournals.org/)

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References

31) M. Okada, Neural Networks **9** (1996), 1429.