\( \mathcal{N} = 4 \) Chern-Simons Theories and Wrapped M-Branes in Their Gravity Duals

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We investigate a class of \( \mathcal{N} = 4 \) quiver Chern-Simons theories and their gravity duals. We define the group of fractional D3-brane charges in a type IIB brane setup taking account of D3-brane creation due to the Hanany-Witten effect, and confirm that it agrees with the 3-cycle homology of the dual geometry, which describes the charges of fractional M2-branes, i.e., M5-branes wrapped on 3-cycles. The relation between the fractional brane charge and the torsion of the 3-form potential field is partially established. We also discuss the duality between baryonic operators in the Chern-Simons theories and M5-branes wrapped on 5-cycles in the dual geometries. The degeneracy and the conformal dimension of the operators are reproduced on the gravity side. We also comment on the relation between wrapped M2-branes and monopole operators. The baryonic operators we consider are not gauge-invariant. We argue that gauge invariance cannot be imposed on all the operators corresponding to wrapped M-branes in the \( \text{AdS}_4/\text{CFT}_3 \) correspondence.

\section{Introduction}

Recently, three-dimensional Chern-Simons matter systems with various numbers of supersymmetries have attracted great interest for describing the low-energy effective theories of M2-branes in various backgrounds. Bagger, Lambert, and Gustavsson\(^1\)\textendash\(^5\) proposed an \( \mathcal{N} = 8 \) Chern-Simons theory as a model for multiple M2-branes. This model (the BLG model) is based on an interesting mathematical structure, a so-called 3-algebra. The consistency condition (the fundamental identity) associated with the 3-algebra is very restrictive, and there is only one example of superconformal theory based on the BLG model, which describes two M2-branes in certain orbifolds.\(^6\),\(^7\)

Since this proposal, many works on superconformal Chern-Simons theories have appeared. In Ref.\(^8\), a new class of Chern-Simons matter systems that possess \( \mathcal{N} = 4 \) supersymmetry was constructed. These systems were generalized in Ref.\(^9\) by introducing new matter multiplets (twisted hypermultiplets).

A theory describing multiple M2-branes in the 11-dimensional flat spacetime was first proposed by Aharony, Bergman, Jafferis, and Maldacena.\(^10\) Their model (the ABJM model) is a \( U(N) \times U(N) \) Chern-Simons theory at level \( (k, -k) \) and possesses \( \mathcal{N} = 6 \) supersymmetry. They show that the model describes multiple M2-branes in the orbifold \( \mathbb{C}^4/\mathbb{Z}_k \). If we take \( k = 1 \), this space becomes \( \mathbb{C}^4 \), and the supersymmetry is expected to be enhanced to \( \mathcal{N} = 8 \) in a nontrivial way. Also see

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We can realize the ABJM model by using a type IIB brane configuration consisting of D3-branes, one NS5-brane, and one \((k, 1)\)-fivebrane.\(^{10}\) By increasing the number of 5-branes, this is generalized to quiver Chern-Simons theories described with circular quiver diagrams.\(^{15}\) If we introduce only two kinds of fivebranes with appropriate directions, the Chern-Simons theory possesses \(\mathcal{N} = 4\) supersymmetry.\(^{16}\) In this case, the corresponding geometry is a certain orbifold of \(\mathbb{C}^4.\)\(^{15}\) See also Refs. \(^{17}\) and \(^{18}\) for orbifolds of the ABJM model. If we introduce fivebranes with three or more different charges, the supersymmetry is at most \(\mathcal{N} = 3\), and we generally obtain curved hyper-Kähler geometries.\(^{19}\)

All these theories are conformal and have gravity duals.\(^{20}\) The purpose of this paper is to investigate some aspects of the gravity duals of \(\mathcal{N} = 4\) Chern-Simons theories.

One aspect considered is about fractional branes. Fractional branes in the ABJM model were investigated in Ref. 21. It was suggested that in the gravity dual, fractional branes are realized as the torsion of the 3-form potential in the orbifolds \(S^7/Z_k\). In this paper we generalize this result to a more general orbifold \(M_{p,q,k} = (S^7/(\mathbb{Z}_p \oplus \mathbb{Z}_q))/\mathbb{Z}_k\) associated with \(\mathcal{N} = 4\) Chern-Simons theories. We find that the homology \(H_3\) of this orbifold is also pure torsion and confirm that it agrees with the group of fractional brane charges in the type IIB brane configuration obtained by taking account of the Hanany-Witten effect.

We also discuss baryonic operators and monopole operators. The gauge symmetry of the \(\mathcal{N} = 4\) Chern-Simons theory is

\[
G = \left( \prod_{I=1}^{n} U(N_I) \right) /U(1)_d = G_{SU} \times G_B, \tag{1.1}
\]

where \(U(1)_d\) is the diagonal \(U(1)\) subgroup that does not couple to any bi-fundamental fields, and \(G_{SU}\) and \(G_B\) are defined by

\[
G_{SU} = \prod_{I=1}^{n} SU(N_I), \quad G_B = \left( \prod_{I=1}^{n} U(1)_I \right) /U(1)_d. \tag{1.2}
\]

The abelian part \(G_B = U(1)^{n-1}\) is called the baryonic symmetry. In the case of four-dimensional quiver gauge theories, the baryonic symmetry is often treated as a global symmetry because in the infrared limit it decouples from the system. In this paper, we treat this part of the gauge group in a similar way. Namely, when we later define baryonic operators, we require gauge invariance with respect to only the \(G_{SU}\) part of the gauge group. Unlike the four-dimensional case, \(G_B\) does not decouple even in the infrared limit, and thus the baryonic operators we discuss in this paper are not gauge-invariant operators. In the case of the ABJM model, the baryonic symmetry \(G_B = U(1)\) is spontaneously broken by the vacuum expectation values of the dual photon field, and we can define gauge-invariant baryonic operators by multiplying appropriate functions of the dual photon field.\(^{22}\) This, however, does not work for \(n \geq 3\).
Despite the gauge variance of baryonic operators, we identify them with M5-branes wrapped on 5-cycles in $M_{p,q,k}$ and confirm that the conformal dimension and multiplicity of the operators are reproduced on the gravity side in the same way as Ref. 23), in which the baryonic operators\(^{24}\) in the Klebanov-Witten theory\(^{25}\) were investigated. This may seem contrary to the usual AdS/CFT dictionary, which relates only gauge-invariant operators to their counterparts on the gravity side. We discuss this point after mentioning the relation between monopole operators and wrapped M2-branes. In three-dimensional spacetime, local operators generally carry magnetic charges. A special class of monopole operators constructed with the dual photon field carry the magnetic charge of the diagonal $U(1)$ gauge group. We thus call them diagonal monopole operators. When $n \geq 3$ there is a greater variety of monopole operators in addition to the diagonal ones. We propose that such nondiagonal monopole operators correspond to M2-branes wrapped on 2-cycles in the internal space.

The rest of this paper is organized as follows. In the next section we summarize the field contents, symmetries, and the moduli space of $\mathcal{N} = 4$ Chern-Simons theories. In §3, we determine the group of fractional branes for $k = 1$ using the type IIB setup, and in §4 we reproduce this group as the homology on the M-theory side. We discuss the correspondence between baryonic operators in Chern-Simons theories and M5-branes wrapped on 5-cycles for $k = 1$ in §5. The analysis in §§3–5 is generalized to $k \geq 2$ in §6. In §7 we again discuss the relation between baryonic operators and wrapped M5-branes. We show that in the type IIB setup, $N$ open strings representing constituent bifundamental quarks can be continuously deformed into a D3-brane disk, which is dual to a wrapped M5-brane. In §8, we discuss the relation between fractional branes and the torsion of the 3-form potential in M-theory. In §9, we comment on the relation between wrapped M2-branes and non-diagonal monopole operators, and explain why we do not impose $G_B$ gauge invariance on baryonic operators. We summarize our results in the final section.

§2. $\mathcal{N} = 4$ Chern-Simons theories

Let us consider an $\mathcal{N} = 4$ supersymmetric Chern-Simons theory with gauge group $(1\cdot 1)$. This theory has the same number of vector multiplets and bifundamental hypermultiplets, and is represented by a circular quiver diagram. The size of the gauge groups $N_I$ may depend on the vertices.

A hypermultiplet $H_I$ contains two complex scalar fields that belong to a doublet of $SU(2)$ R-symmetry. The R-symmetry Spin$(4) = SU(2)^2$ of the $\mathcal{N} = 4$ theory

\[
H_{i-1} \quad H_i \quad H_{i+1} \quad H_{i+2}
\]

\[
U(N_{i-1})_{i-1} \quad U(N_i)_{i} \quad U(N_{i+1})_{i+1}
\]

Fig. 1. Part of a circular quiver diagram of an $\mathcal{N} = 4$ supersymmetric Chern-Simons theory.
Table I. Global symmetries of the $\mathcal{N} = 4$ Chern-Simons theory. A certain combination of $U(1)_{b}$ and $U(1)_{d}$ acts on the hypermultiplets in the same way as a certain $U(1)$ subgroup of gauge symmetry. The two $U(1)$ groups, however, act on the dual photon field differently. For this reason, they are different from each other.

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<td>$U(1)_b$</td>
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includes two $SU(2)$ factors, and correspondingly there are two kinds of hypermultiplets: untwisted and twisted hypermultiplets.\(^9\) We denote the two $SU(2)$ groups by $SU(2)_A$ and $SU(2)_B$, and adopt the convention that scalar components of untwisted and twisted hypermultiplets are nontrivially transformed by $SU(2)_A$ and $SU(2)_B$, respectively. We denote these scalar fields by $h^\alpha_I$ and $\psi^\alpha_I$. (Undotted and dotted indices denote $SU(2)_A$ and $SU(2)_B$ ones, respectively.) Fermions are transformed in the opposite way from the scalar fields. The theory also has two $U(1)$ global symmetries as shown in Table I.

In the type IIB brane system, which consists of D3-branes, NS5-branes, and $(k,1)$-fivebranes, these hypermultiplets arise from the open strings stretched between two adjacent intervals of D3-branes. We use the same index $I$ for fivebranes and hypermultiplets. The two kinds of hypermultiplets correspond to the two different charges of fivebranes. Let us define the number $s_I$ associated with hypermultiplets, which is 0 for untwisted hypermultiplets and 1 for twisted hypermultiplets. The RR charge of the fivebrane associated with the $I$th hypermultiplet is $k s_I$, and the boundary interaction of D3-branes ending on fivebranes induces the Chern-Simons terms\(^{26,27}\)

\[
S = \sum_I k_I \frac{1}{4\pi} \int \text{tr} \left( A_I dA_I + \frac{2}{3} A^3_I \right),
\]

(2.1)

where the level of the $U(N_I)_I$ gauge group coupling to the hypermultiplets $H_I$ and $H_{I+1}$ is given by

\[
k_I = k(s_{I+1} - s_I).
\]

(2.2)

In the following, we refer to the integer $k$ simply as the “level” of the theory.

The moduli space of this theory is analyzed in Ref. 15. We obtain the background geometry of M2-branes as the Higgs branch moduli space of the theory with $N_I = 1$. When $k = 1$, it is the product of two four-dimensional orbifolds,

\[
\mathcal{M}_{p,q} = C^2/Z_p \times C^2/Z_q,
\]

(2.3)

where $p$ and $q$ are the numbers of untwisted and twisted hypermultiplets, respectively. For later convenience, we introduce complex coordinates $z_i$ ($i = 1, 2, 3, 4$) on which the orbifold group acts by

\[
(z_1, z_2, z_3, z_4) \rightarrow (\omega^m_p z_1, \omega^m_p z_2, \omega^n_q z_3, \omega^n_q z_4) \quad m, n \in \mathbb{Z},
\]

(2.4)
where $\omega_n = e^{2\pi i/n}$. When $k \geq 2$, we have an extra $\mathbb{Z}_k$ orbifold
\begin{equation}
(z_1, z_2, z_3, z_4) \to (\omega_{kp}^m z_1, \omega_{kp}^m z_2, \omega_{kq}^{-m} z_3, \omega_{kq}^{-m} z_4) \quad m \in \mathbb{Z}.
\end{equation}
(2.5)

Thus, the background geometry is
\begin{equation}
\mathcal{M}_{p,q,k} = (\mathbb{C}^2 / \mathbb{Z}_p \times \mathbb{C}^2 / \mathbb{Z}_q) / \mathbb{Z}_k.
\end{equation}
(2.6)

When $N_I = N$, the Higgs branch moduli space is the symmetric product of $N$ copies of the orbifold.

The rotational symmetry group of this manifold is
\begin{equation}
(SU(2) \times U(1))^2,
\end{equation}
(2.7)
which agrees with the global symmetry of the Chern-Simons theory shown in Table I. The R-symmetries $SU(2)_A$ and $SU(2)_B$ act on $\mathbb{C}^2 / \mathbb{Z}_p$ and $\mathbb{C}^2 / \mathbb{Z}_q$, respectively.

To obtain the above orbifold, note that a certain subgroup of $G_B$ is spontaneously broken by the vacuum expectation value of the dual photon field $\tilde{a}$, which is defined by
\begin{equation}
d\tilde{a} = \sum_{I=1}^{n} k_I A_I.
\end{equation}
(2.8)

Under the gauge symmetry $A_I \to A_I + dA_I$ the dual photon field is transformed by
\begin{equation}
\tilde{a} \to \tilde{a} + k_I \lambda_I.
\end{equation}
(2.9)

The dual photon field is a periodic scalar field with period $2\pi$, and the operator $e^{i\tilde{a}}$ carries the $U(1)_I$ charge $k_I$. The moduli space is parameterized by a set of mesonic operators. We define the mesonic operators as $G = G_{SU} \times G_{B}$-invariant operators. By definition, they are neutral with respect to the baryonic symmetry $G_B$. All trace operators are mesonic operators. In addition, we can construct the following mesonic operators:
\begin{equation}
b = e^{-i\tilde{a}} \prod_{a=1}^{p} (h_a)^{k}, \quad \tilde{b} = e^{i\tilde{a}} \prod_{\tilde{a}=1}^{\tilde{q}} (h_{\tilde{a}})^{k},
\end{equation}
when $N = 1$.

We suppress the R-symmetry indices in (2.10). The right-hand side of the first equation in (2.10) has $pk$ $SU(2)_A$ indices, and we take the symmetric part of these indices to define $b$. In terms of the $\mathcal{N} = 2$ terminology, the two scalar fields $h_I^1$ and $h_I^2$ are chiral and antichiral fields, respectively, and thus $b^{1\ldots1}$ and $(b^{2\ldots2})^\dagger$ are chiral operators. Owing to the $SU(2)_A$ symmetry, other components also belong to certain short multiplets. In the same way, $\tilde{b}$ has $qk$ symmetric $SU(2)_B$ indices. When the size of the gauge groups is $N \geq 2$, we should replace the dual photon operators in (2.10) by appropriate monopole operators\cite{22,29,30} which require color indices to make (2.10) gauge-invariant.

If we place a large number of $N$ M2-branes at the tip of the orbifold (2.6) and take account of the back-reaction to the metric, we obtain the dual geometry of this system. It is $AdS_4 \times M_{p,q,k}$, where $M_{p,q,k}$ is the section of the orbifold (2.6) at
\begin{equation}
|z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2 = 1.
\end{equation}
(2.11)
This section is the following orbifold of the 7-sphere:

\[ M_{p,q,k} = (S^7/ (\mathbb{Z}_p \oplus \mathbb{Z}_q))/\mathbb{Z}_k. \] (2.12)

The radii of AdS\(_4\) and \(M_{p,q,k}\) are given by

\[ R_{S^7}^6 = (2R_{AdS_4})^6 = \frac{(2\pi l_p)^6}{2\pi^4} Nkpq. \] (2.13)

The radius of \(M_{p,q,k}\) is that of the covering space \(S^7\). The background metric is

\[ ds^2 = R_{AdS_4}^2 ds_{AdS_4}^2 + R_{S^7}^2 ds_{M_{p,q,k}}^2. \] (2.14)

§3. **Fractional D3-branes**

As we mentioned in the last section, the moduli space of an \(\mathcal{N} = 4\) Chern-Simons theory depends only on the level \(k\) and the numbers \(p\) and \(q\) of the two kinds of hypermultiplets. It does not depend on the order of the two kinds of hypermultiplets in the circular quiver diagrams.

This is similar to the situation in the elliptic models of four-dimensional \(\mathcal{N} = 1\) supersymmetric gauge theories. Such theories are generalizations of Klebanov-Witten theory\(^25\) and can be described by type IIA brane systems, which consist of D4-branes wrapped along \(S^1\) and NS5-branes intersecting with the D4-branes. In this brane configuration, NS5-branes are classified into two groups according to their direction. Let us call these NS5-branes with different directions A-branes and B-branes. On the D4-branes a four-dimensional gauge theory, which is described by a circular quiver diagram, is realized. If the numbers of A- and B-branes are \(p\) and \(q\), respectively, the Coulomb branch moduli space of the gauge theory is the symmetric product of a generalized conifold \(uv = x^p y^q\)\(^31\) which depends only on \(p\) and \(q\), and is independent of the order of A- and B-branes along \(S^1\). The field theories sharing the same moduli space are related by the Seiberg duality\(^32\) and flow to the same effective theory in the infrared limit. Such a relation between the interchange of branes and the Seiberg duality was first pointed out in Ref. \(^33\).

It is natural to expect that this is also the case for the three-dimensional Chern-Simons theories we are considering here. Such a brane exchange procedure was applied to three-dimensional Chern-Simons theories in Refs. \(^34\) and \(^35\). The important difference between this duality and the four-dimensional one is that in the three-dimensional case the brane exchange process generally generates new D3-branes owing to the Hanany-Witten effect (Fig. 2)\(^36\).

The purpose of this section is to classify theories that give the same moduli space. We assume that two theories realized by brane systems related by a continuous deformation are dual to each other and flow to the same infrared fixed point. We identify such theories, and the problem considered here is the amount of variety of inequivalent theories that exist for given \(p\), \(q\), and \(k\). We study the \(k = 1\) case first, and later discuss the generalization to \(k \geq 2\).

To realize \(k = 1\) theories, we use D3, NS5, and (1,1) fivebranes. The set of these three kinds of branes is equivalent to the set of D3, NS5, and D5-branes up
to a certain $SL(2, \mathbb{Z})$ duality transformation. Thus, here we use the latter set of branes. The directions of these three kinds of branes are shown in Table II. The D3-branes are wrapped on the compactified direction $x^6$. The $p$ NS5-branes and $q$ D5-branes intersect with the D3-brane worldvolume and divide $S^1$ into $n = p + q$ intervals. We label the intersection points by $I = 1, 2, \ldots, n$ in order along $S^1$. We emphasize that when we use $I$ as a label of fivebranes, it represents the position of each fivebrane along $S^1$. In other words, $I$ is used to label the slots in which we can place fivebranes. We use $a, b = 1, \ldots, p$ and $\dot{a}, \dot{b} = 1, \ldots, q$ to label NS5-branes and D5-branes, respectively, and the location of each fivebrane is specified using the one-to-one map $(a, \dot{a}) \rightarrow I$. We call the choice of this map the “frame”.

We denote the number of D3-branes in the interval between the $I$th and $(I + 1)$th fivebranes by $N_I$. $N_I$ gives the size of each $U(N)$ factor in the gauge group in (1.1). Let us define $m_I$ as the number of D3-branes emanating from the $I$th fivebrane. By definition, $N_I$ and $m_I$ are related by

$$m_I = N_I - N_{I-1}. \quad (3.1)$$

Because the $m_I$ are invariant under the overall shift $N_I \rightarrow N_I + c$, $(c \in \mathbb{Z})$, we cannot uniquely determine $N_I$ from $m_I$. This degree of freedom represents integral D3-branes wrapping around the whole $S^1$. We here focus only on the fractional brane charges and use $m_I$ to represent D3-brane distributions.

In general, the numbers $m_I$ change when the order of fivebranes is changed by continuous deformations. For this reason, to specify the brane configuration, we need not only to give $m_I$ but also to specify the frame, i.e., the order of the fivebranes. In the following we assume that we choose a particular frame.

With a fixed frame, a D3-brane distribution is specified by the vector

$$(m_1, m_2, \ldots, m_p | m_\dot{1}, m_\dot{2}, \ldots, m_\dot{q}). \quad (3.2)$$

We call this vector “charge vector”. By definition, the components of a charge vector

![Fig. 2. Example of D3-brane creation process. (a) is an initial configuration consisting of an NS5-brane and a $(k,1)$-fivebrane. If the $(k,1)$-brane is moved to the other side of the NS5-brane as shown in (b), $k$ D3-branes are created.](image-url)
must satisfy the constraint
\[ \sum_{a=1}^{p} m_a + \sum_{\dot{a}=1}^{\dot{q}} m_{\dot{a}} = 0. \] (3.3)

The set of charge vectors whose components are constrained by (3.3) forms the group
\[ \Gamma = \mathbb{Z}^{p+q-1}. \] (3.4)

We should not regard the group \( \Gamma \) as the group characterizing the conserved charge of fractional D3-branes because D3-brane distributions corresponding to different elements of \( \Gamma \) may be continuously deformed to one another. We should regard the charges of such brane configurations as the same.

Let us move a NS5-brane \( a \) in the positive direction along \( S^1 \) until it returns to the original position. This process does not change the frame, but it changes the charge vector. When the NS5-brane passes through a D5-brane \( \dot{b} \), \( m_a \) decreases by one and \( m_{\dot{b}} \) increases by one. (Here we assume \( k = 1 \).) When the NS5-brane returns to the original position, the charge vector changes by
\[ \mathbf{v}_a = (0, \ldots, -q, \ldots, 0|1, \ldots, 1) = -q \mathbf{e}_a + \sum_{b=1}^{\dot{q}} \mathbf{e}_b \in \Gamma, \] (3.5)

where \( \mathbf{e}_a \) (\( \mathbf{e}_b \)) is the unit vectors whose \( a \)th (\( \dot{b} \)th) component is 1. Note that \( \mathbf{e}_a \) and \( \mathbf{e}_b \) themselves are not elements of \( \Gamma \) because they do not satisfy the constraint (3.3). Similarly, if we move a D5-brane \( \dot{a} \) around the \( S^1 \), the charge vector changes by
\[ \mathbf{w}_{\dot{a}} = (1, \ldots, 1|0, \ldots, -p, \ldots, 0) = \sum_{b=1}^{p} \mathbf{e}_b - p \mathbf{e}_{\dot{a}} \in \Gamma. \] (3.6)

When we identify configurations that are continuously deformed to one another, these vectors should be identified with 0. Therefore, the group describing the charge of fractional branes is the quotient group \( \Gamma/H \), where \( H \) is the subgroup of \( \Gamma \) generated by the vectors \( \mathbf{v}_a \) and \( \mathbf{w}_{\dot{a}} \). This is given by
\[ \Gamma/H = (\mathbb{Z}_p^{q-1} \oplus \mathbb{Z}_q^{p-1} \oplus \mathbb{Z}_{pq})/(\mathbb{Z}_p \oplus \mathbb{Z}_q). \] (3.7)

In the rest of this section, we explain how this expression of the quotient group is obtained.

As we mentioned above, the \( p+q \) vectors \( \mathbf{e}_a \) and \( \mathbf{e}_{\dot{a}} \) are not elements of \( \Gamma \). Let us choose a basis of \( p+q-1 \) linearly independent vectors in \( \Gamma \). We take the following vectors:
\[ \mathbf{f}_a = \mathbf{e}_a - \mathbf{e}_p, \quad (a = 1, \ldots, p-1) \] (3.8)
\[ \mathbf{g}_{\dot{a}} = \mathbf{e}_{\dot{a}} - \mathbf{e}_q, \quad (\dot{a} = \dot{1}, \ldots, (q-1)) \] (3.9)
\[ \mathbf{h} = \mathbf{e}_p - \mathbf{e}_q. \] (3.10)
We can easily check that these vectors span $\Gamma$. To obtain (3.7), we define the subgroup $H' \subset H$ generated by the following elements in $H$:

\begin{align*}
v_p - v_a &= q f_a, \quad (a = 1, \ldots, p - 1) \\
w_{\dot{q}} - w_{\dot{a}} &= pg_{\dot{a}}, \quad (\dot{a} = \dot{1}, \ldots, (q - 1)) \
-pv_p + qw_{\dot{q}} - \sum_{\dot{b}=1}^{\dot{q}} w_{\dot{b}} &= pqh.
\end{align*}

We can easily show that

\[ \Gamma/H' = \mathbb{Z}^{q-1}_p \oplus \mathbb{Z}^{p-1}_q \oplus \mathbb{Z}_{pq}, \quad H/H' = \mathbb{Z}_p \oplus \mathbb{Z}_q, \]

and the relation $\Gamma/H = (\Gamma/H')/(H/H')$ gives (3.7).

§4. Fractional M2-branes

In this section we reproduce the quotient group (3.7) as the 3-cycle homology of the internal space $M_{p,q} := M_{p,q,1}$ of the dual geometry for $k = 1$.

We recall how the geometry is obtained as a dual configuration from the type IIB brane system in Table II. We first perform a T-duality transformation along $x^6$ and then lift the system to the M-theory configuration. As a result we have the configuration shown in Table III. The two kinds of fivebranes are mapped to purely geometric objects: Kaluza-Klein (KK) monopoles. In general, $Q$ coincident KK monopoles are described as an $A_{Q-1}$-type orbifold, and the geometry shown in Table III is the product of $A_{p-1}$ and $A_{q-1}$ singularities. This product space is just the orbifold $\mathcal{M}_{p,q}$ in (2.3).

Fractional M2-branes are realized as M5-branes wrapped on 3-cycles in $M_{p,q}$. The homologies $H_i(M_{p,q}, \mathbb{Z})$ are given by

\[ H_0 = \mathbb{Z}, \quad H_1 = 0, \quad H_2 = \mathbb{Z}^{p+q-2}, \quad H_3 = (\mathbb{Z}^{q-1}_p \oplus \mathbb{Z}^{p-1}_q \oplus \mathbb{Z}_{pq})/(\mathbb{Z}_p \oplus \mathbb{Z}_q), \]

\[ H_4 = 0, \quad H_5 = \mathbb{Z}^{p+q-2}, \quad H_6 = 0, \quad H_7 = \mathbb{Z}. \]

The relevant homology $H_3(M_{p,q}, \mathbb{Z})$ is pure torsion, and it coincides with the group of fractional D3-branes studied in the previous section. In the following we explain how the $H_3$ group in (4.1) is obtained.

For the purpose of considering cycles in $M_{p,q}$, it is convenient to represent $M_{p,q}$ as a $T^2$ fibration over $B = S^5$. This fibration is defined in the following way. We

| Table III. Dual M-theory geometry. Shrinking cycles are denoted by “s”. |
|-----------------|---|---|---|---|---|---|---|---|---|
| M2-branes       | o | o | o |   |   |   |   |   |   |
| $q$ KK monopoles | o | o | o | s | o | o | o | o |   |
| $p$ KK monopoles | o | o | o | o | o | o | o | s |   |

*1 We obtained these homologies by carefully searching for cycles in the manifold. We have checked their consistency with the Poincaré duality and the Mayer-Vietoris exact sequence.
Fig. 3. Orbifold represented as a fibration over the segment $0 \leq t \leq 1$.

introduce a real coordinate $0 \leq t \leq 1$ by rewriting (2.11) as

$$|z_1|^2 + |z_2|^2 = t, \quad |z_3|^2 + |z_4|^2 = 1 - t. \quad (4.2)$$

At a general value of $t$, this defines two 3-spheres, and the orbifold action (2.4) causes them to become Lens spaces $L_p$ and $L_q$. The manifold $M_{p,q}$ is represented as the $L_p \times L_q$ fibration over the segment $0 \leq t \leq 1$. Each of the Lens spaces $L_p$ and $L_q$ can be represented as a $S^1$ fibration over a 2-sphere. For $L_p$, which is rotated by the $SU(2)_A$ R-symmetry, we refer to the base manifold and the fiber as $S^2_A$ and the $\alpha$-cycle, respectively. We also define $S^2_B$ and the $\beta$-cycle for the other Lens space $L_q$, which is rotated by $SU(2)_B$ (Fig. 3). Owing to the $\mathbb{Z}_p \oplus \mathbb{Z}_q$ orbifolding, the periods of the $\alpha$- and $\beta$-cycles are $2\pi/p$ and $2\pi/q$, respectively. If we combine $S^2_A$, $S^2_B$, and the segment parameterized by $t$, they form a 5-sphere $B = S^5$. We can regard the orbifold $M_{p,q}$ as a $T^2$ fibration over $B$.

At $t = 0$, which defines $S^2 \subset B$, the Lens space $L_q$ shrinks and so does the $\alpha$-cycle. Similarly, on $S^2 \subset B$ with $t = 1$ the $\beta$-cycle shrinks. These $S^2$ are linked to each other in $B$. By blowing up the singularities, these $S^2$ split into $p$ and $q$ $S^2$,\(^1\) which we call $x_a$ ($a = 1, \ldots, p$) and $y_{\dot{a}}$ ($\dot{a} = \dot{1}, \ldots, \dot{q}$), respectively (Fig. 4). We can follow the IIB/M duality to verify that each locus corresponds to the fivebrane with the same index.

Three-cycles in $M_{p,q}$ can be represented as $T^2$ fibrations over segments in the base manifold $B = S^5$. There are three types of segments connecting two loci of a degenerate fiber (Fig. 4). We denote a segment connecting a point in $x_a$ and a point in $x_b$ by $[x_a, x_b]$. We similarly define $[y_{\dot{a}}, y_{\dot{b}}]$ and $[x_a, y_{\dot{b}}]$. We also adopt the notation

$$S^\alpha, S^\beta, S^{\alpha\beta} \subset M_{p,q} \quad (4.3)$$

\(^1\) We blow up the singularities simply to make the cycles the well-defined. When we later compute the volume of 5-cycles, we consider the orbifold limit.
for the manifold obtained by combining a subset $S \subset B$ and fibers indicated as superscripts. $S^{\alpha \beta}$ is the $T^2$ fibration over $S$. $S^{\alpha \beta}$ can be regarded as a $\beta$-fibration over a certain base manifold, which is isomorphic to $S^\alpha$. $S^\alpha$ is a global section in this fiber bundle. Therefore, $S^\alpha$ can be defined only when the $\beta$-cycle fibration over $S$ has a global section. Similarly, we can define $S^\beta$ when the $\alpha$-cycle fiber has the trivial topology over $S$.

With this notation, we can represent 3-cycles generating $H_3$ as

$$[x_a, x_b]^{\alpha \beta}, \quad [y_a, y_b]^{\alpha \beta}, \quad [x_a, y_b]^{\alpha \beta}. \quad (4.4)$$

Because the $\alpha$-cycle or the $\beta$-cycle shrinks at the endpoints of the segments, these are closed 3-cycles. The topology of $[x_a, x_b]^{\alpha \beta}$ and $[y_a, y_b]^{\alpha \beta}$ is $S^2 \times S^1$, and that of $[x_a, y_b]^{\alpha \beta}$ is $S^3$.

These 3-cycles are not linearly independent. There are combinations of cycles that can be unwrapped. Let us consider

$$\sum_{a=1}^p [x_a, y_b]^{\alpha \beta}. \quad (4.5)$$

This union of 3-cycles can be unwrapped in $M_{p,q}$. This can be shown by considering a 4-chain whose boundary is (4.5). Such an “unwrapping chain” is constructed in the following way. Because $\pi_2(S^5) = 0$, there is a three-dimensional disk $D^3 \subset B$ whose boundary is $y_b$ (the gray disk in Fig. 5). We call this $Y_b$. (We also define $X_a$
in the same way for $x_a$.) This disk intersects once with every $x_a\ (a = 1, \ldots, p)$. Let $\bar{Y}_b$ be the subset of $Y_b$ obtained by removing segments connecting these intersecting points and $y_b$ (the segments in Fig. 5) from the disk.

$$\bar{Y}_b = Y_b \setminus \sum_{a=1}^{p} [x_a, y_b]. \quad (4.6)$$

Because $\bar{Y}_b$ is contractible, we can define $\bar{Y}_b^\beta$. The boundary of the manifold $\bar{Y}_b^\beta$ is

$$\partial \bar{Y}_b^\beta = \sum_{a=1}^{p} [x_a, y_b]^\beta. \quad (4.7)$$

Before we explain this relation, let us first consider the Hopf fibration of $S^3$ as a simple example. Using the Hopf fibration, $S^3$ is described as an $S^1$ fibration over $S^2$. Let $(\theta, \phi)$ be the polar coordinates of the base $S^2$. The first Chern class of this fiber bundle is 1; thus, we cannot globally define the coordinate of the fiber. We cover the base $S^2$ using two patches, a north patch $S^2 \setminus S$ and a south patch $S^2 \setminus N$, where $S$ and $N$ are the south pole ($\theta = \pi$) and north pole ($\theta = 0$), respectively. Let $0 \leq \psi_N < 2\pi$ and $0 \leq \psi_S < 0$ be the fiber coordinates defined in the north and south patch, respectively. These are glued by the relation $\psi_N = \psi_S + \phi$. Owing to the nonvanishing first Chern class, we cannot take a global section in this fiber bundle. To define sections, we need to remove at least one point (a 0-cycle) from the base $S^2$. Let us remove the north pole. Then, we can cover the remaining part of the base using the south patch $S^2 \setminus N$. We can define, for example, the section

$$0 < \theta \leq \pi, \quad 0 \leq \phi < 2\pi, \quad \psi_S = 0. \quad (4.8)$$

We denote this section as $Z_N$. At the boundary $\theta = 0$ of the south patch $S^2 \setminus N$, this section wraps the fiber $S^1$:

$$\partial Z_N = N^\eta, \quad (4.9)$$

where $\eta$ denotes the fiber $S^1$. This becomes obvious if we use the coordinate $\psi_N$, which is suitable for descriptions around the north pole. The boundary is given by

$$\theta = 0, \quad 0 \leq \phi < 2\pi, \quad \psi_N = \phi. \quad (4.10)$$

This boundary winds once along the fiber $\eta$. This result makes sense from the fact that the homology $H_1(S^3)$ vanishes. Any 1-cycle on $S^3$ can be unwrapped and represented as the boundary of a 2-chain.

Let us return to the case of the $T^2$ fibration over $Y_b$. The $\beta$-cycle is not involved in the derivation of (4.7), and thus we initially neglect it. For simplicity, we consider the case with $p = 1$. Then, we have only one $x_a$. Let us define the standard polar coordinates $(r, \theta, \phi)$ in $Y_b$ so that $r = 1$ and $r = 0$ correspond to $y_b$ and the intersection with $x_a$, respectively. Each shell defined by a fixed $r$ with $0 < r < 1$ is $S^2 \subset Y_b$. This sphere is denoted by $S^2_A$ in Fig. 3, and the $\alpha$-cycle fibration over $S^2_A$ gives the Hopf fibration of $S^3$. The argument in the previous paragraph can be
applied for each $S^3$. We choose the intersection of $S^2_A$ and the segment $[x_a, y_b]$ as the north pole $N$ on $S^2_A$. For each $0 \leq r \leq 1$ we have $N^\alpha$, the $\alpha$-cycle fibration over the north pole $N$, as the boundary of $S^2_A \setminus N$. By collecting these for every $0 \leq r \leq 1$, we obtain the two-dimensional $[x_a, y_b]^\alpha$ as the boundary of $\bar{Y}_b = Y_b \setminus [x_a, y_b]$. Precisely speaking, we also have the boundary $y_b^\alpha$, the $\alpha$-cycle fibration over $y_b$, at $r = 1$. This part, however, does not survive if we take the $\beta$-cycle into account because the $\beta$-cycle shrinks on $y_b$. Then, we obtain $[x_a, y_b]^\alpha \beta$ as the boundary of $\bar{Y}_b^\beta$. For general $p$, each segment $[x_a, y_b]$ generates a boundary 3-cycle and we obtain the relation (4.7).

By exchanging the roles of $X_a$ and $Y_b$, we can also show

$$
\partial \bar{X}_a^\alpha = \sum_{b=1}^{\hat{q}} [x_a, y_b]^\alpha \beta.
$$

(4.11)

From (4.7) and (4.11), we obtain the homology relation

$$
\sum_{a=1}^{p} [x_a, y_b]^\alpha \beta = \sum_{b=1}^{\hat{q}} [x_a, y_b]^\alpha \beta = 0.
$$

(4.12)

To clarify the relation between the type IIB framework and the M-theory framework, we define the formal basis $x_a$ and $y_b$ and rewrite cycles as $[x_a, x_b]^\alpha \beta = x_a - x_b$ and so forth. A general superposition of cycles, which is depicted as a junction in $B$, can be written as the linear combination

$$
j = \sum_{a=1}^{p} m_a x_a + \sum_{\hat{a}=1}^{\hat{q}} m_{\hat{a}} y_{\hat{a}},
$$

(4.13)

where the coefficients must satisfy the constraint (3.3). We obtain a one-to-one correspondence between the 3-cycles in $M_{p,q}$ and the D3-brane distributions in the type IIB framework by simply identifying the coefficients in (4.13) with the components of the charge vector (3.2). Via this isomorphism the boundaries (4.7) and (4.11) correspond to $w_b$ and $v_a$, respectively, the generators of $H$, and the relation (4.12) defines the homology $H_3$ as the same coset group $\Gamma/H$ in (3.7).

\section{Five cycles and baryonic operators}

In this section, we discuss the relation between M5-branes wrapped on 5-cycles and baryonic operators in the $\mathcal{N} = 4$ Chern-Simons theory. For the case of the ABJM model, such analysis was performed in Ref. 22), and the conformal dimension and multiplicity of baryonic operators were reproduced on the gravity side. We extend the results to $\mathcal{N} = 4$ Chern-Simons theories.

As in the previous section, we consider the $k = 1$ case. As given in (4.1), the 5-cycle homology of $M_{p,q}$ is

$$
H_5(M_{p,q}, \mathbb{Z}) = \mathbb{Z}^{p+q-2},
$$

(5.1)
and when \( p + q \geq 3 \), there exist nontrivial cycles on which M5-branes can be wrapped. If we represent \( M_{p,q} \) as the \( T^2 \) fibration over \( B = S^5 \), the 5-cycles can be written as the \( T^2 \) fibrations over 3-disks.

\[
\Omega_a := X^\alpha_a \quad \text{and} \quad \Omega_{\bar{a}} := Y^{\alpha\bar{a}}. \tag{5.2}
\]

These generate the homology \( H_5(M_{p,q}, \mathbb{Z}) \).

The number of cycles in (5.2) is larger than the dimension of \( H_5(M_{p,q}, \mathbb{Z}) \) by two, and there should be two relations among the cycles in (5.2). Indeed, we have the following homology relations:

\[
\sum_{a=1}^p \Omega_a = \sum_{\bar{a}=1}^q \Omega_{\bar{a}} = 0. \tag{5.3}
\]

As performed in §4 for 3-cycles, we can give these linear combinations as the boundaries of unwrapping 6-chains. We define a submanifold \( \bar{B} \subset B \) by

\[
\bar{B} = B \setminus \left( \sum_{a=1}^p X^a + \sum_{\bar{a}=1}^q Y^\bar{a} \right). \tag{5.4}
\]

If we could draw \( S^2 \) enclosing \( x_a \) in \( \bar{B} \), the \( \alpha \)-cycle fiber would have a nontrivial twist on \( S^2 \). However, no such \( S^2 \) exists in \( \bar{B} \) because we removed the disks \( X_a \). Thus, the \( \alpha \)-cycle fiber over \( \bar{B} \) has the trivial topology and we can define global sections. Similarly, owing to the removal of \( Y_{\bar{a}} \), the \( \beta \)-cycle fiber also has the trivial topology. Because there is a global section associated with the \( \alpha \)-cycle over \( \bar{B} \), the manifold \( \bar{B}^\beta \) is well-defined, and its boundary is

\[
\partial \bar{B}^\beta = \sum_{a=1}^p X^\alpha_a. \tag{5.5}
\]

We also obtain

\[
\partial \bar{B}^\alpha = \sum_{\bar{a}=1}^q Y^\alpha_{\bar{a}}. \tag{5.6}
\]

As a result, we obtain the relations (5.3).

What are the corresponding baryonic operators on the gauge theory side? A natural guess is that these 5-cycles are dual to the following operators in the Chern-Simons theory:

\[
\Omega_a \leftrightarrow B^{\alpha_1\alpha_2\cdots\alpha_N}_a = \epsilon_{i_1\cdots i_N} e^{j_1\cdots j_N} h^\alpha_{a\ j_1} \cdots h_{a\ j_N}^{\alpha_N}, \tag{5.7}
\]

\[
\Omega_{\bar{a}} \leftrightarrow B^{\alpha_1\bar{\alpha}_2\cdots\bar{\alpha}_N}_{\bar{a}} = \epsilon_{\bar{i}_1\cdots\bar{i}_N} e^{\bar{j}_1\cdots\bar{j}_N} h^\alpha_{\bar{a}\ \bar{j}_1} \cdots h_{\bar{a}\ \bar{j}_N}^{\alpha_N}. \tag{5.8}
\]

Each of these operators is charged under the baryonic symmetry \( G_B \) and cannot be decomposed into mesonic operators, which are \( G_B \)-neutral. However, the products

\[
\prod_{a=1}^p B_a, \quad \prod_{\bar{a}=1}^q B_{\bar{a}} \tag{5.9}
\]
carry the same baryonic charge as $e^{iN\tilde{a}}$ and $e^{-iN\tilde{a}}$, respectively, and by multiplying by an appropriate power of the operator $e^{i\tilde{a}}$, we can construct neutral operators with respect to the baryonic symmetries. This strongly indicates that they can be decomposed to the mesonic operators as

$$e^{-iN\tilde{a}} \prod_{a=1}^{p} B_a \sim b^N, \quad e^{iN\tilde{a}} \prod_{\tilde{a}=1}^{\hat{q}} B_{\tilde{a}} \sim \tilde{b}^N.$$  

(5.10)

This decomposability corresponds to the homology relation (5.3) among the 5-cycles.

As a nontrivial verification of the duality, let us compare the mass of the wrapped M5-branes and the conformal dimension of the operators. According to the standard AdS/CFT dictionary, the conformal dimension $\Delta$ of an operator and the mass $M$ of the corresponding object are related by $\Delta = R_{\text{AdS}4} M$. In the case of an M5-brane wrapped on $\Omega_a$, this relation becomes

$$\Delta = R_{\text{AdS}4} T_{M5} \frac{R_{S^7}}{2 \pi^3} \text{Vol}(\Omega_a) = \frac{Npq}{2 \pi^3} \text{Vol}(\Omega_a),$$  

(5.11)

where $\text{Vol}(\Omega_a)$ is the volume of the 5-cycle $\Omega_a$ in $M_{p,q}$ with radius 1, and to obtain the last expression we used (2.13) with $k = 1$ and the M5-brane tension $T_{M5} = 2\pi/(2\pi l_p)^6$. Let us calculate the volume of the 5-cycle $\Omega_a$. This 5-cycle, which is represented as a fiber bundle over the segment $0 \leq t \leq 1$, is illustrated as the shaded region in Fig. 6. The radii of the two 3-spheres defined by (4.2) are $r_1 = t^{1/2}$ and $r_2 = (1 - t)^{1/2}$, respectively. The cross section at $t$ is $S^1 \times S^2 \times S^1$ with radii $r_1/p$, $r_2/2$, and $r_2/q$, respectively.\(^\ast\) Hence the volume of the 5-cycle is

$$\text{Vol}(\Omega_a) = \int_{t=0}^{t=1} ds \left(\frac{2\pi r_1}{p}\right) \times \left(\frac{4\pi}{2}\right)^2 \times \left(\frac{2\pi r_2}{q}\right) = \frac{\pi^3}{pq},$$  

(5.12)

\(^\ast\) It is known that when a unit $S^3$ is represented by the $S^1$ fibration over $S^2$, the radii of $S^1$ and $S^2$ are 1 and 1/2, respectively.
where \( ds \) is the line element with respect to the parameter \( t \), computed as

\[
ds^2 = dr_1^2 + dr_2^2 = \frac{1}{4t(1-t)}dt^2.
\]

(The volume (5.12) is simply \( \text{Vol}(S^5)/pq \) because the 5-cycles considered here are orbifolds of a large \( S^5 \) in \( S^7 \).) We obtain the same result for 5-cycles \( Y_{\alpha} \). By substituting this into (5.11) we obtain

\[
\Delta = \frac{1}{2}N,
\]

which agrees with the conformal dimension of the baryonic operators (5.7) and (5.8). (5.14) is consistent with the result of more general analysis in Ref. 37) for generic toric tri-Sasakian manifolds.

The degeneracy of the baryonic operators is explained in the same way as the Klebanov-Witten theory. The collective coordinates of the 5-cycle \( \Omega_a \) are the coordinates in the transverse direction \( S^2_p \), now which \( SU(2)_A \) acts as a rotation. The 7-form flux in the background acts as a magnetic field on \( S^2_p \) with a flux of \( N \). Therefore, the effective theory of the corrective coordinates is the theory of a charged particle in \( S^2_p \) with \( N \) units of magnetic flux. The ground states of the particle are the \( N+1 \) states at the lowest Landau level belonging to the spin \( N/2 \) representation of \( SU(2)_A \). This degeneracy agrees with that of the baryonic operators \( B_a \). In the same way, we can explain the degeneracy of \( B_\dot{a} \) as that of the lowest Landau level of a charged particle in the transverse direction \( S^2_B \).

\[\text{§6. Generalization to } k \geq 2\]

In this section we generalize the analysis in the previous sections to the case of \( k \geq 2 \).

Let us first consider fractional D3-brane charges in the type IIB brane setup. We can realize the Chern-Simons theory at level \( k \) by replacing D5-branes by \((k,1)\) fivebranes. We can again represent the distributions of D3-branes using charge vectors (3.2) with their components constrained by (3.3). The only difference from the \( k = 1 \) case is that when a \((k,1)\)-brane and an NS5-brane pass through each other, not one but \( k \) D3-branes are generated. As a result, the vectors (3.5) and (3.6) are multiplied by the extra factor \( k \). Namely, we should replace the subgroup \( H \) with \( kH \) and \( k\mathbf{w}_a \), and the quotient group becomes

\[
\Gamma/(kH) = \Gamma/(kH') \rightarrow (H/H') = (Z_{kp}^{-1} \oplus Z_{kp}^{-1} \oplus Z_{kpq})/(Z_p \oplus Z_q).
\]

On the other hand, the homologies \( H_i(M_{p,q,k}, Z) \) are

\[
H_0 = Z, \quad H_1 = Z, \quad H_2 = Z^{p+q-2}, \quad H_3 = (Z_{kp}^{-1} \oplus Z_{kp}^{-1} \oplus Z_{kpq})/(Z_p \oplus Z_q), \quad H_4 = 0, \quad H_5 = Z^{p+q-2} \oplus Z, \quad H_6 = 0, \quad H_7 = Z.
\]

We find that the homology \( H_3 \) is again identical to (6.1). Let us construct the homology \( H_3(M_{p,q,k}, Z) \) more explicitly. When the level \( k \) is greater than 1, we have
an additional $\mathbb{Z}_k$ factor in the orbifold group. As shown in (2.5), the generator of $\mathbb{Z}_k$ shifts both $\alpha$- and $\beta$-cycles by $1/k$ of their periods. Because the two cycles do not shrink at the same time at any point, this action does not generate fixed points. The $\mathbb{Z}_k$ identification in the $T^2$ fiber generates new cycles that not integral linear combinations of $\alpha$ and $\beta$. They are multiples of

$$\gamma = \frac{1}{k}(\alpha - \beta).$$

As a result, the 2-cycle defined as the product of $\alpha$ and $\beta$ is not the fundamental $T^2$ but its multiple $kT^2$. Thus, the cycles (4.4) are decomposed into $k$ copies of the following elementary cycles:

$$[x_a, x_b]^\gamma, \quad [y_a, y_b]^\gamma, \quad [x_a, y_b]^\gamma.$$ (6.4)

Owing to this fact, the boundaries of unwrapping 4-chains (4.11) and (4.7) are replaced by

$$\partial \bar{X}_a^\alpha = k \left( -q x_a + \sum_{b=1}^{\hat{q}} y_b \right), \quad \partial \bar{Y}_a^\beta = k \left( \sum_{b=1}^{p} x_b - p y_a \right),$$ (6.5)

where we define the formal basis $x_a$ and $y_a$ by $[x_a, x_b]^\alpha = x_a - x_b$ and so forth. These trivial 3-cycles $k(-qx_a + \sum_{b=1}^{\hat{q}} y_b)$ and $k(\sum_{b=1}^{p} x_b - p y_a)$ precisely correspond to the vectors $k\mathbf{v}_a$ and $k\mathbf{w}_a$ respectively and thus the homology $H_3$ becomes isomorphic to the quotient $\Gamma/kH$ in (6.1).

Next, let us consider the relation between baryonic operators and the 5-cycle homology $H_5$ for $k \geq 2$. In the case of $k \geq 2$, the generators of the homology in (5.2) should be replaced by

$$\Omega_a = X_a^{\gamma \alpha}, \quad \Omega_a = Y_a^{\gamma \alpha},$$ (6.6)

and the M5-branes wrapped on these generating cycles are identified with the baryonic operators $B_a$ and $B_a$. We can again easily verify that the volume of the 5-cycles correctly reproduces the conformal dimension $\Delta = N/2$. The $p+q$ generators (6.6) are not linearly independent, and we can take $\bar{B}^\alpha$, $\bar{B}^\beta$, and $\bar{B}^\gamma$ as unwrapping 6-chains, which give the relation among these generators. Their boundaries are

$$\partial \bar{B}_a^\alpha = \sum_{\hat{a}=1}^{\hat{q}} Y_{\hat{a}}^{\alpha \beta} = k \sum_{\hat{a}=1}^{\hat{q}} \Omega_{\hat{a}},$$ (6.7)

$$\partial \bar{B}_a^\beta = \sum_{a=1}^{p} X_a^{\alpha \beta} = k \sum_{a=1}^{p} \Omega_a,$$ (6.8)

$$\partial \bar{B}_a^\gamma = \sum_{a=1}^{p} X_a^{\gamma \beta} + \sum_{\hat{a}=1}^{\hat{q}} Y_{\hat{a}}^{\gamma \alpha} = \sum_{a=1}^{p} \Omega_a + \sum_{\hat{a}=1}^{\hat{q}} \Omega_{\hat{a}}.$$ (6.9)

Namely, these linear combinations of 5-cycles are in the trivial element of the homology $H_5$. Upon dividing the group $\mathbb{Z}^{p+q}$, generated by the $p+q$ bases, $\Omega_a$ and $\Omega_{\hat{a}}$
by its subgroup $Z^2$ generated by the above boundaries, we obtain the $H_5$ homology in (6.2).

On the field theory side, the linear dependence of the 5-cycles is interpreted as the decomposability of the products of the baryonic operators into the mesonic operators. The first two boundaries, (6.7) and (6.8), correspond to the products of the $B_a$ and the $\tilde{B}_a$, respectively, and are decomposed into the $N$th power of the operators defined in (2.10),

$$e^{-iN\tilde{a}} \prod_{a=1}^p B_a^k \sim b^N, \quad e^{iN\tilde{a}} \prod_{\tilde{a}=1}^q \tilde{B}_{\tilde{a}}^k \sim \tilde{b}^N. \quad (6.10)$$

The third boundary (6.9) corresponds to the product of all the $p+q$ baryonic operators, and it can be decomposed into trace operators.

$$\prod_{a=1}^p B_a^k \prod_{\tilde{a}=1}^q \tilde{B}_{\tilde{a}}^k \sim \left[ \text{tr} \left( \prod_{a=1}^p h_a \prod_{\tilde{a}=1}^q h_{\tilde{a}} \right) \right]^N. \quad (6.11)$$

The degeneracy of baryonic operators for $p+q \geq 3$ is again reproduced in the same way as for the $k=1$ case. For the case of $p=q=1$ (the ABJM model), we need a special treatment because the global symmetry (2.7) is enhanced to $SU(4) \times U(1)$ and the motion of collective coordinates is treated as a point particle in $SU(4)/(SU(3) \times U(1))$. This is considered in Ref. 22) and the correct multiplicity is obtained.

§7. Quark-baryon transition

In §5, we studied the relation between wrapped M5-branes and baryonic operators $B_I$. We can relate them more directly by using the IIB/M duality explained in §4. Using this duality, we can easily see that an M5-brane wrapped on $\Omega_I$ is dual to a D3-brane disk ending on fivebrane $I$, and as we explain below, the D3-brane disk can be continuously deformed to $N$ open strings corresponding to the constituent bifundamental quarks. (Similar transitions in different brane systems are also considered in Refs. 39) and 44).)

Before we explain the deformation, we comment on a relevant fact about flux conservation on the worldvolume of a D3-brane ending on an NS5-brane. The $U(1)$ gauge field $A$ on an NS5-brane electrically couples to the endpoints of D-strings on the NS5-brane. This is also the case for the magnetic flux $f = da$ on D3-branes, which can be regarded as D-strings dissolved in the D3-brane worldvolume. This coupling is described as the action

$$S = \frac{1}{2\pi} \oint_{\partial D3} A \wedge f. \quad (7.1)$$

By integrating by parts, this is rewritten as

$$S = \frac{1}{2\pi} \oint_{\partial D3} a \wedge F, \quad (7.2)$$
which implies that the flux $F = dA$ on the NS5-brane behaves as an electric charge on the boundary of the D3-brane coupled by the gauge field $a$. If the D3-brane worldvolume is compact, the electric flux conservation requires that the total electric charge vanishes. If the integral of flux $F$ over the D3-brane boundary is $2\pi N$, we need $N$ strings ending on the D3-brane worldvolume to compensate for the boundary charge. This is also the case for a D3-brane ending on a $(k, 1)$ fivebrane.

Bearing this fact in mind, we can show that $N$ open strings and a D3-brane disk can be continuously deformed to each other. In the following we consider three sets of D3-branes and to distinguish them we name them as follows:

- $X$ – the $N$ coincident D3-branes between fivebranes $I$ and $I - 1$.
- $Y$ – the $N$ coincident D3-branes between fivebranes $I$ and $I + 1$.
- $D$ – a D3-brane disk whose boundary is $S^2$ on fivebrane $I$.

We here assume that $N_I = N_{I-1} = N$. Let us start from a D3-brane disk $D$ whose boundary is $S^2$ on the fivebrane $I$ enclosing the boundaries of both $X$ and $Y$ (Fig. 7(a)). Although these boundaries carry magnetic charges coupled by $A$, their charges cancel each other, and the net flux passing through the boundary $\partial D$ is zero. There are no open strings ending on $D$.

We move the disk so that $\partial Y$, the boundary of $Y$, becomes outside $\partial D$. When $\partial Y$ passes through $\partial D$, the flux through $\partial D$ increases by $N$, and $N$ open strings stretched between $Y$ and $D$ are generated so that the total electric charge on the disk is canceled (Fig. 7(b)).

If we keep moving the disk so that $\partial X$ also becomes outside the boundary $\partial D$, the flux through the boundary decreases by $N$, and this time $N$ open strings stretched between $D$ and $X$ are generated. The two sets of $N$ strings can be connected to get off $D$, and we obtain $N$ open strings connecting $X$ and $Y$ (Fig. 7(c)). The disk can be annihilated without any obstructions.

If $m_I = N_I - N_{I-1} \neq 0$, the D3-brane disk $D$ in Fig. 7(a) has $m_I$ strings attached on it. This corresponds to the fact that we cannot define $SU(N_{I-1}) \times SU(N_I)$-invariant operators such as (5.7) and (5.8) due to the mismatch of the number of indices. The $m_I$ open strings attached on the D3-brane disk correspond to the $m_I$ fundamental or $-m_I$ antifundamental indices, which are not contracted.
§8. Three-form torsion and fractional branes

In this section, we relate the fractional brane charge and integrals of the 3-form field on 3-cycles. Let us consider a process in which the number of fractional branes changes. The fractional brane charge \( Q \in H_3 \) affects the 3-form field \( C_3 \) and is measured by integrals over the 3-cycles \( \zeta \)

\[
\oint_{\zeta} C_3, \quad \zeta \in H_3.
\] (8.1)

We define the period integral at \( r = r_0 \) between the horizon \( r = 0 \) and the AdS boundary \( r = \infty \). To change the fractional brane charge by \( \Delta Q \), we add an M5-brane wrapped on a 3-cycle \( \Delta Q \in H_3 \) at the AdS boundary and move it to the horizon. When the M5-brane passes through \( r = r_0 \), the period integrals change by

\[
\Delta \oint_{\zeta} C_3 = 2\pi \langle \zeta, \Delta Q \rangle,
\] (8.2)

where \( \langle *, * \rangle \) is the map \( H_3 \times H_3 \rightarrow U(1) \) the so-called torsion linking form or, simply, the linking number.

The linking number is defined as follows. Let \( s \) be the order of \( \zeta \). Namely, \( s \) is the smallest positive integer such that \( s\zeta \) is homologically trivial. Such an integer always exists because \( H_3 \) is pure torsion. There exists a 4-chain \( D \) such that

\[
s\zeta = \partial D.
\] (8.3)

We define the linking number \( \langle \zeta, \eta \rangle \) of two 3-cycles \( \zeta \) and \( \eta \) by

\[
\langle \zeta, \eta \rangle = \frac{1}{s} \langle \langle D, \eta \rangle \rangle,
\] (8.4)

where \( \langle \langle D, \eta \rangle \rangle \) is the number of intersections of the 4-chain \( C \) and 3-cycle \( \eta \). Because this number changes by integers upon continuous deformation, only the fractional part of the linking number is a topological invariant.

If we move an M5-brane wrapped on the 3-cycle \( \Delta Q \) from the AdS boundary to the horizon, when it passes through \( r = r_0 \), the M5-brane intersects with the 4-chain \( D \) at \( \langle \langle D, \eta \rangle \rangle \) points. In this process, the 4-form flux \( G_4 \) passing through \( D \), including the contribution of Dirac’s stringlike objects, changes by \( 2\pi \langle \langle D, \zeta \rangle \rangle \). Using Stokes’ theorem we obtain the relation (8.2).

For the manifold \( M_{p,q,k} \), from the definition of the linking number, we can easily obtain

\[
k\langle v_a, j \rangle = m_a, \quad k\langle w_{\dot{a}}, j \rangle = m_{\dot{a}},
\] (8.5)

for a general 3-cycle \( j \) in (4.13). The linking numbers among the bases are

\[
\langle x_a, x_b \rangle = -\frac{1}{kq} \delta_{ab}, \quad \langle y_{\dot{a}}, y_{\dot{b}} \rangle = -\frac{1}{kp} \delta_{\dot{a}\dot{b}}, \quad \langle x_a, y_{\dot{b}} \rangle = -\frac{1}{2kpq}.
\] (8.6)

Owing to the constraint (3.3), linking number among the bases is not unique. For example, a constant shift of all the linking numbers in (8.6) does not affect the linking
numbers for 3-cycles that are a linear combination of the bases with the coefficients constrained by (3-3).

By integrating the relation (8.2) and using (8.5), we obtain

\[ m_a - m_0^a = \frac{k}{2\pi} \oint_{v_a} C_3, \quad m_\dot{a} - m_0^{\dot{a}} = \frac{k}{2\pi} \oint_{w_\dot{a}} C_3, \]  

(8.7)

where \( m_a^0 \) and \( m_\dot{a}^0 \) are constants of integration, which cannot be determined from (8.2).

Although gauge transformations can change the period integrals of \( C_3 \), the relation (8.7) determines an element of \( \Gamma/kH \) in a gauge-invariant way if we know \( m_a^0 \) and \( m_\dot{a}^0 \), because a large gauge transformation changes the charge vector by an element of \( kH \).

An important fact is that the constants \( m_a \) and \( m_\dot{a} \) depend on the frame, i.e., the order of the fivebranes. The right-hand sides of the relations (8.7) are defined on the M-theory side and are independent of the frame, while \( m_a \) and \( m_\dot{a} \) on the left-hand side change by multiples of \( k \) when we change the order of the fivebranes. This means that \( m_a^0 \) and \( m_\dot{a}^0 \) depend on the frame, and we cannot simply set them to zero.

To obtain some information about the constants, we use branes corresponding to baryonic operators. We recall that in the type IIB setup, baryonic operators correspond to D3-brane disks ending on fivebranes, and when \( m_I \neq 0 \), they are accompanied by \( m_I \) open strings.

A similar phenomenon occurs on the M-theory side. If there is nontrivial background \( C \)-field, then M5-branes wrapped on 5-cycles are accompanied by M2-branes attached on their worldvolume. Identifying these M2-branes with strings in the type IIB setup, we obtain relations between \( m_I \) and the background \( C \)-field.

Let us consider the flux conservation on M5-branes and how it relates to the background \( C \)-field and the M2-branes attached on it. The 2-form field \( b_2 \) on M5-branes couples with the field strength \( G_4 \) in the bulk by the coupling

\[ S = \frac{1}{2\pi} \int_{M5} b_2 \wedge G_4. \]  

(8.8)

This implies that the flux behaves as charge on M5-branes. On the worldvolume of an M5-brane wrapped on a 5-cycle, the total charge coupled by \( b_2 \) must cancel owing to the flux conservation. This implies that the cohomology class of the total charge,

\[ \left[ \frac{1}{2\pi} G_4 - \delta(\partial M2) \right] \in H^4(\Omega_I, \mathbb{Z}), \]  

(8.9)

must be trivial. \( \delta(\partial M2) \) is the 4-form delta function with support on the boundaries of M2-branes. By the Poincaré duality, this is equivalent to

\[ [g] = [\partial M2] \in H_4(\Omega_I, \mathbb{Z}), \]  

(8.10)

where \( g \) is the 1-cycle Poincaré dual to the flux \( (2\pi)^{-1} G_4 \). The homologies \( H_i(\Omega_a, \mathbb{Z}) \) in the 5-cycle are given by

\[ H_0 = \mathbb{Z}, \quad H_1 = \mathbb{Z}_k, \quad H_2 = \mathbb{Z}^{q-1}, \quad H_3 = \mathbb{Z}^{q-1} \oplus \mathbb{Z}_k, \quad H_4 = 0, \quad H_5 = \mathbb{Z}. \]  

(8.11)
The homologies in $\Omega_a$ are obtained by replacing $q$ in (8.11) by $p$. Because $H_1(\Omega_I, \mathbb{Z}) = \mathbb{Z}_k$ is pure torsion we can rewrite (8.10) in terms of the linking form $H_3 \times H_1 \to U(1)$ as

$$\frac{1}{2\pi} \oint_{\zeta} C_3 = \langle \zeta, \partial M2 \rangle,$$

(8.12)

where $\zeta$ is the generator of the torsion subgroup of $H_3(\Omega_I, \mathbb{Z})$: $\zeta = v_a$ for $\Omega_a$ and $\zeta = w_\dot{a}$ for $\Omega_\dot{a}$. If we identify $m_I$ strings ending on a D3-brane disk with an M2-brane wrapped on $m_I \gamma$, where $\gamma$ is the generator of $H_1(M_{p,q,k}, \mathbb{Z}) = H_1(\Omega_I, \mathbb{Z})$, (8.12) can be rewritten as

$$m_a = \frac{k}{2\pi} \oint_{v_a} C_3, \quad m_\dot{a} = \frac{k}{2\pi} \oint_{w_\dot{a}} C_3 \mod k.$$

(8.13)

This means that

$$m_a^0 = m_\dot{a}^0 = 0 \mod k.$$

(8.14)

This fixes only the frame-independent part of $m_a^0$ and $m_\dot{a}^0$. Although in the case of $p = q = 1$ this reproduces the result in Ref. 21 for the ABJM model, this is not sufficient to establish the relation between the fractional brane charge and the 3-form torsion for $p + q \geq 3$. We leave this problem for future works.

## §9. Wrapped M2-branes and monopole operators

The correspondence between the Kaluza-Klein modes of massless fields in the internal manifold and the primary operators in the corresponding boundary CFT is one of the most important claims of the AdS/CFT correspondence.

Such a correspondence for the ABJM model is discussed in Refs. 10, 40 and 41. For more general $\mathcal{N} = 2$ quiver gauge theories, which describe M2-branes in toric Calabi-Yau 4-folds, the relation between the holomorphic monomial functions, which are specified by the charges of toric $U(1)$ symmetries, and mesonic operators consisting of bifundamental fields was proposed in Ref. 42). In the reference, a simple prescription to establish a concrete correspondence between Kaluza-Klein modes and mesonic operators is given by utilizing brane crystals.\textsuperscript{42–44) When this method was proposed, it had not yet been realized that the quiver gauge theories are actually quiver Chern-Simons theories. After the importance of the existence of Chern-Simons terms was realized, this proposal was confirmed\textsuperscript{45–47) for a special kind of brane crystal, which can be regarded as an “M-theory lift” of brane tilings.\textsuperscript{48–50)}

In three-dimensional spacetime, local operators generally carry magnetic charges. Such operators are called monopole operators. In the correspondence between primary operators in three-dimensional CFT and Kaluza-Klein modes, monopole operators play an important role. The results in Refs. 48–50) indicate that the set of primary operators corresponding to the supergravity Kaluza-Klein modes includes only a special kind of monopole operators, which we call “diagonal” monopole operators. Diagonal monopole operators carry only the diagonal $U(1)$ magnetic charges and are constructed by combining dual photon fields and chiral matter fields. Concrete examples of diagonal operators have already been given in (2.10). Because the
canonical conjugate of the dual photon field is the diagonal $U(1)$ field strength $F_D$, the operator $e^{im\tilde{a}}$ shifts the flux $F_D$ by $m$.

We now consider a general Abelian quiver $\mathcal{N} = 2$ Chern-Simons theory. We label the vertices by $a$ and denote the corresponding gauge group by $U(1)_a$. Let us consider a monopole operator with magnetic charges $m_a \in \mathbb{Z}$. The diagonal monopole operator $e^{im\tilde{a}}$ carries the same magnetic charge $m_a = m$ for all the $U(1)_a$ gauge groups. The gauge invariance of the operator requires the Gauss law constraint

$$m_a k_a + Q_a = 0,$$  \hspace{1cm} (9.1)

where $Q_a$ is the $U(1)_a$ electric charge carried by matter fields included in the monopole operator. This guarantees the invariance of the operator under the gauge symmetry (2.9). By summing over all $a$, we obtain the constraint

$$\sum_a m_a k_a = 0.$$  \hspace{1cm} (9.2)

Therefore, monopole operators are labeled by $n - 1$ independent magnetic charges. One of them is the diagonal monopole charge, which corresponds to a certain component of Kaluza-Klein momentum in the internal space (the D-particle charge from the type IIA perspective).

How can the interpretation of the other $n - 2$ magnetic charges be interpreted? It is natural to identify them with the charges of M2-branes wrapped on 2-cycles.

Let us return to the $\mathcal{N} = 4$ Chern-Simons theory studied in this paper, which is a special case of $\mathcal{N} = 2$ quiver Chern-Simons theories. The 2-cycle homology of the corresponding internal space $M_{p,q,k}$ is

$$H_2(M_{p,q,k}, \mathbb{Z}) = \mathbb{Z}^{p+q-2},$$  \hspace{1cm} (9.3)

and the Betti number coincides with the number of independent magnetic charges of nondiagonal monopole operators.

We now explain why we did not impose $G_B$ gauge invariance on the baryonic operators. First, we recall the reason why symmetry groups that act on wrapped branes are usually regarded as global symmetries. Consider $AdS_{d+1}$ with the metric

$$ds^2 = \frac{R^2}{z^2}((dx^\mu)^2 + dz^2), \hspace{1cm} \mu = 1, \ldots, d,$$  \hspace{1cm} (9.4)

and let $A_\nu(x^\mu, z)$ be a $U(1)$ gauge field coupling to wrapped branes. In accordance with Ref. 51) we consider the Euclidian AdS space, in which $z$ is the radial coordinate such that the AdS boundary is at $z = 0$. Let us assume the asymptotic behavior of the vector field to be

$$A_\nu \propto z^\Delta.$$  \hspace{1cm} (9.5)

For the convergence of the Euclidian action, $\Delta$ must satisfy the inequality

$$\frac{d}{2} - 2 < \Delta.$$  \hspace{1cm} (9.6)
Using the equation of motion $d \ast dA = 0$ we obtain the asymptotic behavior of the gauge field

$$A_\nu(x^\mu, z) = a_\nu(x^\mu) + z^{d-2}b_\nu(x^\mu).$$

(9.7)

On the AdS boundary we need to impose a boundary condition that fixes either $a_\nu(x^\mu)$ or $b_\nu(x^\mu)$. When $d \geq 4$, only the second term in (9.7) is allowed by (9.6) and the boundary condition $a_\nu(x^\mu) = 0$ must be imposed. Then the gauge field asymptotically vanishes near the boundary, which is the reason why the symmetry is global in the boundary CFT.

On the other hand, when $d = 3$, both terms in (9.7) satisfy the inequality (9.6), and we can choose either $a_\nu(x^\mu) = 0$ (Dirichlet) or $b_\nu(x^\mu) = 0$ (Neumann) as the boundary condition. Indeed, these two boundary conditions first appeared in Ref. 52) and are used in Ref. 53) to construct a pair of Chern-Simons theories that are “S-dual” to each other. Let us take the Neumann boundary condition. In this case, the boundary value of the gauge field $a_\nu(x^\mu) = A_\nu(x^\mu, z = 0)$ does not vanish and is dynamical in the sense that it is path integrated. Thus, we can regard this as a gauge field in the boundary CFT, and the wrapped branes coupled by $A_\nu$ should also be charged objects in the boundary CFT. Because the Dirichlet and Neumann boundary conditions are exchanged by the duality transformation of the gauge field, both kinds of operators, corresponding to electric and magnetic particles in the AdS$_4$, cannot be gauge-invariant.

In the case of our M-theory background, the gauge fields $A^i = A^i_\nu dx^\nu$ coupling to wrapped M5-branes and $\tilde{A}_i = A_{i\nu} dx^\nu$ coupling to wrapped M2-branes are defined by

$$C_6 = \sum_{i=1}^{p+q-2} \omega_i \wedge A^i, \quad C_3 = \sum_{i=1}^{p+q-2} \omega^i \wedge \tilde{A}_i,$$

(9.8)

where $C_3$ and $C_6$ are the 3- and 6-form potential fields, which are dual to each other, and $\omega_i$ and $\omega^i$ are cohomology bases of $H^3(M_{p,q,k}; \mathbb{Z}) = \mathbb{Z}^{p+q-2}$ and $H^2_{\text{free}}(M_{p,q,k}; \mathbb{Z}) = \mathbb{Z}^{p+q-2}$, respectively. Because $A^i$ and $\tilde{A}_i$ are electric-magnetic-dual to each other, it is impossible to impose the Dirichlet boundary condition on all of them, and consequently some of the wrapped branes inevitably correspond to gauge-variant operators. This is the reason why we did not require the baryonic operators to be $G_B$ gauge-invariant. Although it may be possible to consider an S-dual framework in which wrapped M5-branes correspond to gauge-invariant operators, we would then have to relate wrapped M2-branes to gauge-variant operators.

§10. Conclusions

In this paper, we investigated some aspects of the gravity dual of $\mathcal{N} = 4$ quiver Chern-Simons theories.

We confirmed that the group of fractional brane charges, which is obtained by the analysis of the Hanany-Witten effect in the type IIB brane configuration, is isomorphic to the homology $H_3(M_{p,q,k}; \mathbb{Z})$. We also established a relation between the fractional brane charge and the torsion of the 3-form field up to the frame
dependent-constants. To determine the constant part, more detailed analysis is needed.

We also discuss the duality between baryonic operators in the Chern-Simons theory and M5-branes wrapped on 5-cycles in $M_{p,q,k}$. We defined baryonic operators that carry $G_B$ charges, and found that the homology group $H_5(M_{p,q,k}, \mathbb{Z})$ is consistent with the decomposability of the products of baryonic operators into mesonic operators on the field theory side. We also found that the conformal dimension of baryonic operators is consistent with the mass of the wrapped M5-branes. The degeneracy of the baryonic operators was explained as the degeneracy of the ground states for the collective motion of the wrapped M5-branes.

We also commented on the relation between nondiagonal monopole operators and wrapped M2-branes. The 2-cycle Betti number $b_2$ of the internal manifold $M_{p,q,k}$ was found to coincide with the number of independent magnetic charges of nondiagonal monopole operators.

We did not impose gauge invariance on baryonic operators. In §9 we showed that some of the wrapped M2-branes and wrapped M5-branes inevitably correspond to gauge-variant operators in the boundary CFT.

There are many questions that should be studied. The extension of our analysis to more general quiver Chern-Simons theories with smaller supersymmetry is one unsolved problem. Moduli spaces of $N = 2$ supersymmetric quiver Chern-Simons theories were studied in Refs. 28) and 45)–47). For the class of theories described by brane tilings$^{48)–50)}$ (see also Refs. 54) and 55) for reviews) there is a simple prescription to establish the relation between toric data of Calabi-Yau 4-folds and Chern-Simons gauge theories$^{45)–47)}$ It may be interesting to extend our analysis to such a large class of theories.

In general, the dual CFT of toric Calabi-Yau 4-folds cannot be described by brane tilings. In such a case, brane crystals$^{42)–44)}$ are expected to play an important role. The relation between brane crystals and the dual CFT is not fully understood, and the analysis of homologies and wrapped branes may be helpful for obtaining some information about the dual CFT.

We hope to return to these subjects in the near future.

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