Use of the Generalized Jost Function in Quantum Field Theory

---An extension of the Levinson theorem---

Takeshi KANKI

Institute of Physics, College of General Education
Osaka University, Toyonaka, Osaka Prefecture

(Received July 23, 1962)

The structures of $S$-matrix are investigated in terms of the so-called generalized Jost function. Starting from the partial wave dispersion relation, we can derive the analytic properties of this function in the single-channel scattering and it is shown that there exists a close connection between the analyticity of this function and the generalized Levinson relation. In the case where the one-particle singularity in scattering amplitudes is due to the elementary particle, the corresponding Jost function has a pole, while in the case where this singularity is due to the composite (bound state), the Jost function does not have the pole. Singularities corresponding to the Castillejo-Dalitz-Dyson ambiguity are completely separated. Thus, by making the conjecture that these analyticity in the low-momentum region still remains in the multichannel case, we can extend the Levinson theorem to the multichannel case and to the $S$-matrix theory where the Hamiltonian is not used explicitly. We shall also derive the new representation of $S$-matrix and how the structure of $S$-matrix changes according to each case mentioned above will be clarified.

§ 1. Introduction

One of the most important problems in the elementary particle physics is to know whether a particle is elementary or composite. Many authors discussed this problem but no definite conclusion has yet been drawn. Their discussions are based on the generalized Levinson theorem

$$\delta_l(0) - \delta_l(\infty) = (n_l - m_l)\pi, \quad (1.1)$$

where $\delta_l(k)$ is the scattering phase shift of the $l$-th partial wave for the incident momentum $k$ in the center of mass system, and $n_l$ and $m_l$ are the numbers of discrete levels of the total and the free Hamiltonian, respectively. An elementary particle in the intermediate state of scattering gives the contribution of unit to both the $n_l$ and $m_l$, while the composite particle (bound state) contributes only to the $n_l$. Therefore, if the left-hand side of (1.1) is measurable, the above-mentioned problem can be solved. However, this theorem has so far been proved only in the single channel scattering, while in our field theory infinite many channels open in the high energy limit where we measure the phase shift $\delta_l(\infty)$.

The purpose of this paper is to show the close connection of generalized...
Levinson's relation (1·1) with the analyticity of the so-called generalized Jost function\(^\text{(*)}\) and consequently with the structure of S-matrix. Thus, it is possible to extend the Levinson theorem to the general case where inelastic channels open and to the S-matrix theory where the Hamiltonian is not used explicitly. In § 2, as a preliminary, we shall discuss the analytic properties of S-matrix in the complex k-plane. The generalized Jost function will be introduced in § 3 and the analyticity of this function will be discussed in connection with the analytic properties of S-matrix. We shall see there how the information given by the Levinson theorem is reflected onto the analyticity of the Jost function. Namely, this function has a pole in the case where the particle, appearing in the intermediate state of scattering, is elementary, but the function has not the pole in the case of the composite particle (bound state). Singularities corresponding to the Castillejo-Dalitz-Dyson (C.D.D.) ambiguity\(^\text{(**)}\) are completely separated. In § 4, we shall prove the new integral representation of S-matrix. This representation is just the generalization of that recently obtained by Martin\(^6\) in the case of non-relativistic potential scattering. Of course, we have different representations according to each case mentioned above, that is the case of elementary particle or of composite particle. Some discussions about this point will be given in § 5.

**§ 2. Analyticity of S-matrix**

The analytic continuation of S-matrix into the second Riemann sheet has recently been investigated by many authors.\(^7\)-\(^9\) As a preliminary, we shall briefly review this continuation. Our starting point is the dispersion relation for the l-th partial wave scattering amplitude \(F_l(v),\)

\[
F_l(v) = \sum \frac{g_l}{v + \epsilon_l^2} + \frac{1}{\pi} \int_{-\infty}^{\infty} dv' \frac{\text{Im} F_l(v')}{v' - v} + \frac{1}{\pi} \int_{v}^{\infty} dv' \frac{\text{Im} F_l(v')}{v' - v}, \quad (2·1)
\]

where \(v = k^2\). Thus, the S-matrix

\[
S_l(v) = 1 + 2i\rho(v) F_l(v) \quad (2·2)
\]

with

\[
\rho(v) = \left(\frac{v}{v + \mu^2}\right)^{1/2}
\]

is analytic in the complex \(v\)-plane except for the two branch cuts along the real axis, i.e. \(v \geq 0\) and \(v \leq -\mu^2\), and poles at \(v = -\epsilon_l^2\). Here the branch of \(\rho(v)\) has

\(^{(*)}\) Because we do not deal with the study of the connection of this function with the wave function, this name should not be understood in the strict sense. We use this name only for convenience.

\(^{(**)}\) For simplicity, we shall consider the scattering of two neutral spinless particles with equal mass \(\mu\). The generalization is trivial.
been so chosen as to have \( \rho^*(\nu) = -\rho(\nu^*) \) and \( \rho(\nu + i\varepsilon) \geq 0 \) for \( \nu \geq 0 \). The well-known reality condition can be written as

\[
S_t^*(\nu) = S_t(\nu^*). \tag{2.3}
\]

By virtue of (2.3) and the unitarity condition, we get the relation

\[
S_t^*(\nu + i\varepsilon) = S_t(\nu - i\varepsilon) = \frac{1}{S_t(\nu + i\varepsilon)} \tag{2.4}
\]

for \( \nu \) below the inelastic threshold. Equation (2.4) defines the analytic continuation of \( S_t(\nu - i\varepsilon) \) into the second Riemann sheet, and thus we see that the \( S_t(\nu) \) in the second sheet is just inverse of that in the first (physical) sheet, i.e.

\[
S_t^{II}(\nu) = \frac{1}{S_t(\nu)}. \tag{2.5}
\]

Now let us consider the \( S \)-matrix as a function of the momentum \( k \). The first and second Riemann sheets of \( \nu \)-plane are mapped onto the upper and lower halves of \( k \)-plane, respectively. The relations (2.3) and (2.5) become

\[
S_t^*(k) = S_t(-k^*) \tag{2.6}
\]

\[
S_t(k) = \frac{1}{S_t(-k)}. \tag{2.7}
\]

The location of the singularities of \( S_t(k) \) is illustrated in Fig. 1: Two branch lines along the imaginary axis correspond to the left-hand branch cuts of \( F_t(\nu) \) in both Riemann sheets and the inelastic cuts appear along the real axis; the poles of \( F_t(\nu) \) at \( \nu = -\kappa_i^2 \) become the poles of \( S_t(k) \) at \( k = i\kappa_i \); if \( F_t(\nu) \) shows the resonance, some complex poles must be added to the lower half plane as shown in Fig. 1.

\section*{§ 3. Construction of the generalized Jost function}

We shall construct the generalized Jost function, by starting from the analyticity of \( S \)-matrix in \( k \)-plane and with the aid of the generalized Levinson relation (1.1). Therefore, we consider in this section only the single-channel scattering and assume the existence of the
Hamiltonian. In such a case, the Levinson relation (1.1) is known to be valid. The relation (2.7) suggests us the possibility of the representation

\[ S_t(k) = \frac{f_i(k)}{f_i(-k)}. \]  

(3.1)

We shall further impose the following conditions on the function \( f_i(k) \):

(A) \( f_i(k) \) is analytic in the lower half \( k \)-plane (\( \text{Im} \, k < 0 \));

(B) \( f_i(k) \) tends to unity, when \( |k| \to \infty \);

(C) \( f_i(k) \) has a property

\[ f_i^*(k) = f_i(-k^*). \]  

(3.2)

(D) when the pole of \( S_t(k) \) at \( k = +i\kappa \) is the one-particle singularity due to the elementary particle, \( f_i(k) \) has a pole there, i.e. \( k = +i\kappa \), and when this pole of \( S_t(k) \) corresponds to the singularity due to the bound state, \( f_i(k) \) has zero at \( k = -i\kappa \), and is regular at \( k = +i\kappa \); \( f_i(k) \) has no other zeros in the lower half plane.

The function \( f_i(k) \) is known in the non-relativistic quantum mechanics as the "Jost function".\(^{10}\) The conditions (A)-(C) are the analogues of those of non-relativistic case. (B) is the strong condition, because the condition \( S_t(\infty) = 1 \) requires only \( f_i(k) = f_i(-k) \) for \( |k| \to \infty \). The definition (3.1) allows the multiplicative factor of arbitrary polynomial of \( k^2 \) on \( f_i(k) \). The condition (B) removes this trivial ambiguity of \( f_i(k) \) but, on the other hand, restricts ourselves to the case where \( F_i(\nu) \sim \text{const} \times 1/\nu^\alpha \) (here \( \alpha > 0 \)) for large \( \nu \), as will be seen later. The condition (D) is the point of our paper. We can identify the pole of the left-hand side of Eq. (3.1) with either the pole of the numerator or the zero of the denominator. The reason why we classify the pole into two cases defined in (D) will be discussed later.

Let us explicitly construct \( f_i(k) \), thus proving its existence. With the aid of Eqs. (3.1) and (3.2), it is found that the phase of \( f_i(k) \) on the real axis is equal to the phase shift \( " \delta_t(k) " \) which relates to the \( S \)-matrix through the relation,

\[ S_t(k) = \exp[2i\delta_t(k)]. \]  

(3.3)

By virtue of Eq. (2.6) the phase shift is an odd function of \( k \);

\[ \delta_t(-k) = -\delta_t(k). \]  

(3.4)

Now consider the function

\[ \varphi_t(k) = \ln \left[ \Pi_{i=1}^n \frac{k - i\kappa_i}{k + i\kappa_i} \cdot f_i(k) \right], \kappa_i > 0, \]  

(3.5)

where the factor \( \Pi_{i=1}^n (1/k + i\kappa_i) \) cancels all the possible zeros of \( f_i(k) \) in the lower half plane, that is, according to the condition (D) the products should be taken
Use of the Generalized Jost Function in Quantum Field Theory

over all poles of $S_l(k)$ corresponding to the bound states. The properties (A) and (B) tell us that the function $\varphi_l(k)$ is analytic everywhere in the lower half $k$-plane and vanishes at infinity. Then we have a dispersion relation

$$\varphi_l(k-i\varepsilon) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{d\kappa'}{\kappa' - k - i\varepsilon} \text{Im} \frac{\varphi_l(k')}{\kappa'} \quad (3.6)$$

for real $k$. The imaginary part can be written as

$$\text{Im} \varphi_l(k) = \text{Im} \left[ \ln f_i(k) + \sum_{i=1}^{N} \{ \ln(k - i\varepsilon_i) - \ln(k + i\varepsilon_i) \} \right]$$

$$= \delta_i(k) + \sum_{i=1}^{N} \{ \arg(k - i\varepsilon_i) - \arg(k + i\varepsilon_i) \}. \quad (3.7)$$

Here the argument of the function $(k - i\varepsilon_i/k + i\varepsilon_i)$ is multivalued and we choose the branch as shown by Fig. 2. Therefore, the second term of the right-hand side of (3.7) vanishes at infinity on the real axis. When $\delta_i(k) \sim 1/k^2$ for $|k| \to \infty$, the integral (3.6) converges and we get the following representation for $f_i(k)$ in the lower half $k$-plane,

$$f_i(k) = \prod_{i=1}^{N} \left( \frac{k + i\varepsilon_i}{k - i\varepsilon_i} \right) \exp \left[ -\frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{d\kappa'}{\kappa' - k} \left( \delta_i(k') + \sum_{i=1}^{N} \{ \arg(k' - i\varepsilon_i) - \arg(k' + i\varepsilon_i) \} \right) \right] \quad (3.8)$$

(for $\text{Im} k \leq 0$).

If $\delta_i(k) \sim 1/\ln k$, (3.8) diverges and the subtraction is necessary. Then we get, in this case, $f_i(k) \neq 1$. Thus, we see that condition (B) restricts us to the case where $\Phi(\nu) \sim 1/\nu^2$. It is easily verified that the representation (3.8) satisfies the conditions (A)–(D) in the lower half plane and has a phase $\delta_i(k)$ on the real axis. Now we can continue this function “(3.8)” analytically into the upper half $k$-plane, by using the new analytic function

$$f_i'(k) = S_i(k) f_i(-k), \quad (3.9)$$

Fig. 2. The "arg$(k-i\varepsilon_i/k+i\varepsilon_i)$". The line along the imaginary axis from $-i\varepsilon_i$ to $+i\varepsilon_i$ is the branch line.

4) In this representation, replace the factor $(k-i\varepsilon_i)$ by $(k-i\varepsilon_i)$, where $\varepsilon_i \neq 0$ and $\varepsilon_i > 0$. Then the same result as the representation (3.8) still holds. Therefore, it may be seen that (3.8) is not unique, but that is not the case. Because the function $\phi(k) = (k-i\varepsilon_i/k+i\varepsilon_i)$ has a representation

$$\phi(k) = \exp \left[ -\frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{d\kappa'}{\kappa' - k} \arg(k' - i\varepsilon_i) - \arg(k' + i\varepsilon_i) \right]$$

in the lower half plane. Thus, the two representations of $f_i(k)$ are identical.
which is defined in the upper half \(k\)-plane and coincides with the function (3.8) on the real axis. [See Eqs. (3.2), (3.3) and (3.4).] Thus the existence of the function \(f_i(k)\) we seek has been proved.

Let us consider possible singularities of \(f_i(k)\). Equation (3.9) shows that \(f_i(k)\) has singularities only on the positive imaginary axis, because \(f_i(k)\) is regular in the lower half plane. These are the branch cut from \(+i\mu\) to \(+i\infty\) and the poles corresponding to elementary particles but not bound states. There is a singularity which does not correspond to any singularity of \(S_i(k)\). Such a singularity can occur only at the origin: See the representation (3.8). The numerator of the integrand of (3.8) is an odd function, since \(\delta_i(k)\) is an odd function as shown in (3.4). Then, if this numerator is finite at the origin, (3.8) produces the singularity at the origin. By virtue of our assumption of high energy behaviour of \(F_i(\nu)\), the Levinson theorem (1.1) becomes

\[
\delta_i(0) = (N(l) - M(l)) \pi.
\]

The latter expression can be written as

\[
\delta_i(0) = (N(l) - M(l)) \pi, \quad \text{(3.10)}
\]

by using the number of bound states "\(N(l)\)" and that of the C.D.D. zeros "\(M(l)\)". Combining this with the relation

\[
\lim_{k \to 0 \pm} \arg(k - i\epsilon_i) - \arg(k + i\epsilon_i) = \mp \pi,
\]

which is obtained by virtue of our choice of the branch, and noticing that the summation in the numerator of the integrand of (3.8) is carried out over all bound states of angular momentum \(l\), we find

\[
\lim_{k \to 0 \pm} \sum_{\epsilon_i} \arg(k - i\epsilon_i) - \arg(k + i\epsilon_i) = \mp M(l) \pi. \quad \text{(3.11)}
\]

Thus, the following important result is obtained from (3.8): Our generalized Jost function \(f_i(k)\) has an isolate singularity at the origin, the order of which is proportional to the number of C.D.D zeros of the amplitude \(F_i(\nu)\), i.e. 

\[
f(k) \sim \text{const} \times 1/(E^{2l}(l), \text{ and accordingly, when the C.D.D. zero does not exist, } f_i(k) \text{ is regular at the origin.}
\]

If we rule out the case where the C.D.D. zero exist, then we get a unique representation of \(f_i(k)\), as will be shown in the next section. Now we shall discuss why we think such a function as \(f_i(k)\) satisfies condition (D). The reason is as follows: If the bound state corresponds to the pole of \(f_i(k)\), we can never obtain the unique representation for \(f_i(k)\), because the singularity occurs at the origin and arbitrary constants are introduced concerning this singularity. Next, we consider the case where the pole due to the elementary particle of \(S_i(k)\) corresponds to the zero of \(f_i(k)\). In this case, the constant concerning
Use of the Generalized Jost Function in Quantum Field Theory

the singularity at the origin is determined by the condition that the Jost function is zero at $k = -i\kappa$, but this function coincides with $[(k^2 + \kappa^2)/k^2] \cdot f_1(k)$ and gives the same $S$-matrix as that of $f_1(k)$, where $f_1(k)$ is the function satisfying the condition (D). Therefore, this case is trivial and we do not consider further. We, thus, see that the contents of the generalized Levinson theorem are completely reflected on the analytic structure of the generalized Jost function.

§ 4. The integral representation

Let us carry out the Cauchy integral for the function

$$G_i(k) = \frac{1}{\sqrt{\mu + ik}} \left( f_i(k) - 1 \right)$$

along the path shown in Fig. 3, and we can prove the important integral representation of $S$-matrix. Here, the branch line of the function $\rho(k) = (\mu + ik)^{1/2}$ has been chosen as $i\rho (i\rho + 0) = -i\rho (i\rho - 0) < 0$ for real $\rho \geq \mu$. Thus, it is obtained [for simplicity, we assume $S_i(k)$ has only one pole at $k = i\kappa$]:

(1) The case where the one-particle singularity is due to the elementary particle,

$$f_i(k) = 1 - \frac{\lambda}{ik + \kappa} - \frac{\sqrt{\mu + ik}}{\pi} \int_{\rho}^{\infty} \frac{\omega(p)}{ik + p} \, dp \quad (4\cdot1)$$

and

$$S_i(k) = 1 - \frac{\lambda}{ik + \kappa} - \frac{\sqrt{\mu + ik}}{\pi} \int_{\rho}^{\infty} \frac{\omega(p)}{ik + p} \, dp \quad (4\cdot2)$$

(2) The case where the one-particle singularity is due to the composite particle (the bound state),

$$f_i(k) = 1 - \frac{\sqrt{\mu + ik}}{\pi} \int_{\rho}^{\infty} \frac{\omega(p)}{ik + p} \, dp \quad (4\cdot3)$$

and

$$S_i(k) = 1 - \frac{\sqrt{\mu + ik}}{\pi} \int_{\rho}^{\infty} \frac{\omega(p)}{ik + p} \, dp \quad (4\cdot4)$$

The weight function $\omega(p)$ can be obtained from the knowledge of the discontinuity of $F_i(p)$ across the left-hand cut, that is, we get for the case (2)
Fig. 3. The location of the singularities of \( f_{1}(k) \) and the path of the Cauchy integral.

\[
S_{f}(i\rho - 0) + S_{f}(i\rho + 0) = 2 \left[ \frac{1 + \sqrt{\rho - \mu \cdot \omega(\rho)}}{1 - \sqrt{\rho + \mu \cdot \omega(\rho')}} \right],
\]

where the left-hand side is equal to \( (1 - 2\rho (i\rho + 0) \times \text{Im} F_{1}(i\rho + 0)) \). Write \( \omega(\rho) \) as

\[
\omega(\rho) = H(\rho) W(\rho),
\]

where

\[
H(\rho) = \left[ 1 - 2\rho (i\rho + 0) \text{Im} F_{1}(i\rho + 0) \right] \sqrt{\rho + \mu}.
\]

Then \( W(\rho) \) is determined by the Fredholm type integral equation,

\[
W(\rho) = W_{0}(\rho) - \frac{1}{\pi} \int_{\rho}^{\infty} \frac{H(\rho')}{\rho + \rho'} W(\rho') d\rho' \quad (4.7)
\]

and

\[
W_{0}(\rho) = \frac{-2\rho (i\rho + 0) \text{Im} F_{1}(i\rho + 0)}{H(\rho) \sqrt{\rho - \mu}}. \quad (4.8)
\]

The representation (4.4) just coincides with that recently obtained by Martin in the case of the non-relativistic potential scattering.\(^6\) For the case (1), an equation similar to (4.7) is derived but \( W_{s}(\rho) \) now depends on an arbitrary constant \( \lambda \).

\section{5. Discussion}

When inelastic channels are opened, \( S_{i}(k) \) can be written as,\(^9\)

\[
S_{i}(k) = S_{i}(k) - \frac{g_{i}(k)}{g_{i}(-k)}, \quad (5.1)
\]

where

\[
g_{i}(k) = \exp \left[ i \frac{2}{\pi} \int_{-\infty}^{\infty} dk' \frac{\delta_{i}'(k')}{k' - k} \right] \quad (5.2)
\]

is the contribution from the inelastic cuts and \( \delta_{i}'(k) \) is the imaginary part of the phase shift. \( S_{i}(k) \) has no inelastic cuts and thus, \( S_{i}(k) \) has an analytic structure similar to that of \( S_{i}(k) \) itself in the single channel case. The discussions in the previous sections are based on the validity of the generalized Levinson theorem and we cannot use this theorem in the case where the inelastic
channels are opened, because this theorem has been so far proved only in the single channel case and in the conventional Hamiltonian formalism. However, all of the contents of this theorem are reflected in the analytic structure of the low-momentum region of the generalized Jost function, as we saw in § 3. It should be expected that no such essential changes of the analytic structure in the low-momentum region as some pole vanishes, some zero vanishes or new pole appears, etc., can occur by opening the inelastic channels which correspond to rather high momentum phenomena. Thus, we conjecture that \( S_{\perp}(k) \) in (5.1) also has the representation (4.2) or (4.4), because \( S_{\perp}(k) \) has an analytic structure similar to that of \( S \)-matrix, \( S_{\perp}(k) \), in the single channel scattering and satisfies the unitarity condition like as (2.4), though \( S_{\perp}(k) \) does not. Now the Levinson theorem can be extended to the multi-channel case. Namely, from (5·1) and (5·2), we get

\[
\bar{S}_{\perp}(k) = \exp[2i\bar{\delta}_{\perp}(k)]
\]

and

\[
\bar{\delta}_{\perp}(k) = \delta^R_{\perp}(k) - \frac{2k}{\pi} \lim_{\alpha} \int d\alpha' \frac{\delta_{\perp}^I(k')}{k'^2 - k^2},
\]

where \( \delta^R_{\perp}(k) \) is the real part of the phase shift. When \( \bar{S}_{\perp}(k) \sim 1/k^\gamma \) for \( |k| \rightarrow \infty \), (\( \gamma > 0 \)), the following modified Levinson relation is obtained, after the discussions in § 3 reversely;

\[
\bar{\delta}_{\perp}(0) - \lim_{k \rightarrow \infty} \left[ \delta^R_{\perp}(k) - \frac{2k}{\pi} \lim_{\alpha} \int d\alpha' \frac{\delta_{\perp}^I(k')}{k'^2 - k^2} \right] = (n_1 - m_1)\pi.
\]

Our representations (4·2) and (4·4) clearly show the difference of the structure of \( S \)-matrix arisen as to whether the one-particle singularity is due to the elementary particle or the composite particle. In the conventional dispersion theory, this difference is very unclear. Recently, many authors discussed a possible relation between the position of one-particle singularity and its residue [the coupling constant \( g \) in (2·1)] in the dispersion formalism. However, in the Hamiltonian formalism of the Yukawa type interaction, we have so far considered that the coupling constant is independent of the position of the pole. This apparent gap between these two formalisms can be easily understood in our theory; that is, the case of the Hamiltonian formalism is just the case of (4·2) and in this case, the coupling constant \( g \) is independent of the position of the pole, because of the existence of parameter \( \lambda \). Contrary to this, the case discussed by them corresponds to (4·4) and the coupling constant is determined by (4·4), if \( \omega(\rho) \) is given.
Acknowledgement

The author expresses his sincere thanks to Dr. T. Ogimoto for kindly discussing the problem.

References

6) A. Martin, Nuovo Cimento X. 21 Suppl. 2 (1961), 157. His recent works are summarized in this paper.
9) T. Ogimoto, Prog. Theor. Phys. 27 (1962), 396.