A Quantization of a Spinor Field in the Generalized Hilbert Space

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A quantization of a spinor field is analyzed on the basis of the general Lagrangian, which gives the Dirac equation, in the generalized Hilbert space. The Lagrangian is classified into three: the first one is essentially equivalent to the Dirac one except for sign and is rather docile, the second is a new type and the field considered should be quantized essentially in an indefinite metric space, and the third is the Majorana one. It is also shown that the first field is a mixture of two definite or two indefinite Majorana fields and the second is a mixture of a definite and an indefinite Majorana fields. The expansion of a field in momentum space to get a particle interpretation is rather obvious for the first field but not for the second.

§ 1. Introduction

About ten years ago, Jauch\(^1\) presented rather strange covariant commutation relations of a spinor field, and showed that they have intermediate features between those of the Dirac field and of the Majorana field. Further he asserted that the commutation relations cannot follow from Schwinger’s action principle.\(^2\) Later, Takahashi\(^3\) and Kamefuchi and Tanaka\(^4\) made clear that Jauch’s commutation relations are naturally given by any quantization procedures from a suitable Lagrangian density. However, they restricted their quantization in a positive definite metric space, and showed that Jauch’s commutation relations are the most general ones under the restriction.

However, the restriction to definite metric seems to be inadequate. There often appears particles of indefinite metric as ghost states through interactions of particles of definite metric, which insists on building formalisms in an indefinite metric space from the outset. Indeed Heisenberg has constructed his unified theory in the indefinite Hilbert space by using such states actively. Even in recent quark models of elementary particles, it seems to be more plausible that quarks might be particles of indefinite metric because they should not come out in any real physical processes. Moreover, there have been several attempts to use fields of indefinite metric to eliminate infinity in regularization theories, in which indefinite spinors whose Lagrangians are the Dirac type ones with

\(^1\) The main part of this research was carried out at the Department of Theoretical Physics, University of Liverpool, England.
reversed sign are mixed. It is shown in this paper that there are new types of indefinite spinor other than the above one when the restriction of definiteness of the Hilbert space is ceased. A massless spinor field of this new type, assigned to neutrino, will be discussed on another paper in this journal.

In this paper, we give the most general consideration of the Dirac spinor in the generalized Hilbert space. We deal with the Lagrangian because it determines not only field equations but also covariant commutation relations by a quantization procedure, say Takahashi and Umezawa's or Schwinger’s. The main troubles appear in understanding the particle picture of the fields by quantization.

In § 2, the most general Lagrangian for a spinor field is presented both in a case of a c-number field and of a q-number field. Covariant commutation relations are also given for a q-spinor field. In § 3, the Lagrangian of a q-spinor field is classified and transformations to typical forms are given. In § 4, a gauge transformation of the general Lagrangian is given, and an electromagnetic interaction is introduced by a method of a generalized x-dependent gauge invariance. In § 5, quantization of the field is dealt, and particularly a rather curious expansion of the field in momentum space is made to get particle interpretation. In § 6, some remarks on the new indefinite spinor are given.

§ 2. Lagrangian

We consider a spinor field \(\psi\) which satisfies the Dirac equation

\[
(\gamma_\mu \partial_\mu + \kappa)\psi = 0 ,
\]

where

\[
\gamma_\mu \gamma^\nu + \gamma^\nu \gamma_\mu = 2\delta^\mu_\nu , \quad \gamma^H = \gamma_\mu .
\]

Define associated spinors as follows: an adjoint spinor \(\bar{\psi} = \psi^\dagger \gamma_4\), a charge conjugate spinor \(\psi^C = C \bar{\psi}^T\), and an adjoint of charge conjugate spinor \(\bar{\psi}^C = (C^{-1} \psi)^T\). They satisfy the equations

\[
\bar{\psi} (\gamma_\mu \partial_\mu - \kappa) = 0 , \quad (\gamma_\mu \partial_\mu + \kappa) \psi^C = 0 ,
\]

\[
\bar{\psi}^C (\gamma_\mu \partial_\mu - \kappa) = 0 ,
\]

as a result of Eq. (2.1). Here the matrix \(C\) is defined by

\[
\gamma^\mu = \gamma_\mu \gamma^4 = -C^{-1} \gamma_\mu C ,
\]

\[
C^T = -C , \quad C^H C = 1 ,
\]

and the suffixes \(T, H\) and \(K\) mean transposition, Hermitian conjugation and complex conjugation of a c-number matrix, and \(^\dagger\) means combined Hermitian conjugation of a spinor \(\psi\) not only as a c-number matrix but as a quantum
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operator. Hence, $\psi^\dagger = \psi^\mu$ for a c-number $\psi$.

Define a differential operator $\Gamma = \gamma_\mu \partial_\mu + \kappa = \frac{i}{2} \gamma_\mu (\overrightarrow{\partial}_\mu - \overleftarrow{\partial}_\mu) + \kappa$ which have the properties

$$\Gamma^\tau = C^{-1} \Gamma C, \quad \Gamma^\mu = \gamma^\dagger_\mu \gamma_\mu.$$

(2.8)

Then, there are four bilinear forms which can be candidates of the Lagrangian deriving the Dirac equation (2.1). They are

$$L_1 = -\overline{\psi} \Gamma \psi, \quad L_2 = -\overline{\psi}^\circ \Gamma \psi^\circ,$$

$$L_3 = -\overline{\psi}^\circ \Gamma \psi, \quad L_4 = -\overline{\psi} \Gamma \psi^\circ.$$

(2.9)

c-number theory The Lagrangian is not a matrix but an ordinary number and so $L^\tau$ should be equal to $L$; $L^\tau = L$. We can easily see, using the properties (2.8), that

$$L_1 = L_1^\tau = -L_2, \quad L_2 = L_2^\tau = -L_1,$$

$$L_3 = L_3^\tau = -L_4, \quad L_4 = L_4^\tau = -L_3 = 0.$$

(2.10)

We further assume that the Lagrangian should be real; $L = L^k = L^\mu$ considering $L = L^\tau$. Since $L_1^\mu = L_2$ and $L_2^\mu = L_4$, there remains only one independent bilinear form $L_1$ as a candidate, that is, the Lagrangian is given by

$$\mathcal{L} = -\overline{\psi} (\gamma_\mu \partial_\mu + \kappa) \psi$$

(2.11)

in the c-number theory. This is the ordinary Dirac Lagrangian.

The Majorana theory cannot exist for a c-number spinor, because

$$\mathcal{L} = L_1 = 0$$

(2.12)

under the Majorana requirement $\psi^\circ = \psi$.

q-number theory Bilinear forms are not ordinary numbers but quantum operators, so that they are not necessarily equal to their transposed ones. The Lagrangian is restricted by the following requirement,

Requirement 1. The Lagrangian should be Hermitian.\(^a\)

Since

$$L_1^\dagger = L_1, \quad L_2^\dagger = L_2, \quad L_3^\dagger = L_4, \quad L_4^\dagger = L_3,$$

(2.13)

there are four Hermitian forms

$$L_1, \quad L_2, \quad L_3 = L_3 + L_4, \quad L_4 = i (L_3 - L_4)$$

(2.14)

as candidates of the Lagrangian. The most general Lagrangian is therefore a linear combination of these with the real coefficients $a, b, d$ and $f$:

$$\mathcal{L} = aL_1 + bL_2 + cL_3 + fL_4$$

$$= -a\overline{\psi} \Gamma \psi - b\overline{\psi}^\circ \Gamma \psi^\circ - D\overline{\psi}^\circ \Gamma \psi - D^* \overline{\psi} \Gamma \psi^\circ$$

(2.15)

\(^a\) Since we are considering an indefinite metric quantization, the Hermiticity and unitarity should be understood in the general sense.\(^b\)
Here the matrix element \( D = d + if \), and
\[
\mathcal{Y} = \begin{pmatrix} \phi \\ \phi^c \end{pmatrix}, \quad \overline{\mathcal{Y}} = \langle \overline{\phi}, \overline{\phi}^c \rangle.
\]

In order to get the Euler equation from this Lagrangian by quantum variation \( \{ \mathcal{Y}, \delta \mathcal{Y} \} = \{ \overline{\mathcal{Y}}, \delta \mathcal{Y} \} = 0 \), we must take care of the relation
\[
\mathcal{Y} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathcal{Y}^c = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} C(\overline{\mathcal{Y}})^T,
\]

analogous to the Majorana condition \( \psi = \psi^c \). A variation of the Lagrangian is, then, given by
\[
\delta \mathcal{L} = - \delta \overline{\mathcal{Y}} \begin{pmatrix} a & D^* \\ D & b \end{pmatrix} \mathcal{Y} \mathcal{Y} - \overline{\mathcal{Y}} \begin{pmatrix} a & D^* \\ D & b \end{pmatrix} \delta \mathcal{Y}
\]
\[
= - \delta \overline{\mathcal{Y}} \begin{pmatrix} a & D^* \\ D & b \end{pmatrix} \mathcal{Y} \mathcal{Y} - \overline{\mathcal{Y}} \begin{pmatrix} b & D^* \\ D & a \end{pmatrix} \delta \mathcal{Y}
\]
\[
= - \delta \overline{\mathcal{Y}} \begin{pmatrix} a + b & 2D^* \\ 2D & a + b \end{pmatrix} \mathcal{Y} \mathcal{Y},
\]

and the Euler equation is
\[
- \begin{pmatrix} a + b & 2D^* \\ 2D & a + b \end{pmatrix} \mathcal{Y} \mathcal{Y} = 0.
\]

In the case of \( \Delta = (a + b)^2 - 4DD^* \neq 0 \), Eq. (2.19) becomes \( \mathcal{Y} \mathcal{Y} = 0 \), i.e. the Dirac equations are deduced,
\[
(\gamma^\mu \partial_\mu + \kappa) \psi = 0, \quad (\gamma^\mu \partial_\mu + \kappa) \psi^c = 0.
\]

Now we follow Takahashi and Umezawa's quantization procedure. In our case the differential operator \( A \) of the Euler equation, which is deduced directly from the Lagrangian, is
\[
A = - \begin{pmatrix} a + b & 2D^* \\ 2D & a + b \end{pmatrix} \mathcal{Y},
\]

and the \( R \)-operator defined by \( RA = \square - \kappa^2 \) is
\[
R = - \frac{1}{\Delta} \begin{pmatrix} a + b & -2D^* \\ -2D & a + b \end{pmatrix} (\gamma^\mu \partial_\mu - \kappa).
\]

Then, the commutation relation is given by
\[
\{ \mathcal{Y}(x), \overline{\mathcal{Y}}(x') \} = iRA(x-x'),
\]

or putting \( S(x-x') = (\gamma^\mu \partial_\mu - \kappa) \Delta(x-x') \), it becomes
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\{\psi(x), \overline{\psi}(x')\} = \{\psi^c(x), \overline{\psi}^c(x')\} = -i\frac{a+b}{\Delta} S(x-x'),

\{\psi(x), \overline{\psi}^c(x')\} = i\frac{2D^*}{\Delta} S(x-x'), \quad (2.24)

\{\psi^c(x), \overline{\psi}(x')\} = i\frac{2D}{\Delta} S(x-x').

On the other hand, in the case of \(\Delta=0\), the complex number \(D\) is given by

\[ D = \pm \frac{a+b}{2} e^{i\theta}, \quad (2.25) \]

and the Lagrangian becomes

\[ \mathcal{L} = -\frac{a+b}{2} \overline{\phi}_\pm \Gamma \phi_\pm - \frac{a-b}{2} (\overline{\psi}^c \psi - \overline{\psi} \psi^c), \quad (2.26) \]

where

\[ \phi_\pm = e^{i\theta/2} \phi + e^{-i\theta/2} \phi^c, \quad \phi_- = i(e^{i\theta/2} \phi - e^{-i\theta/2} \phi^c) \quad (2.27) \]

are the Majorana fields: \( \phi_\pm^c = \phi_\pm \).

Now we further add the following requirement to the Lagrangian after Schwinger.\(^*\)

Requirement 2. The Lagrangian should be so given that identical contributions are produced by the terms which differ fundamentally only in the position of field variations.

Considering the variation (2.18), the requirement fixes

\[ a = b. \quad (2.28) \]

Hence in the case of \(\Delta=0\), the Lagrangian (2.26) reduces to

\[ \mathcal{L} = -a \overline{\phi}_\pm \Gamma \phi_\pm, \quad (2.29) \]

i.e. this singular case corresponds to the well-known Majorana theory.

We distinguish the Majorana field by a sign of the coefficient \(a\) of its Lagrangian (2.29), and call a definite Majorana field for \(a>0\), and an indefinite Majorana field for \(a<0\).*

In the general case \(\Delta \neq 0\), the Lagrangian (2.15) becomes

\[ \mathcal{L} = -\overline{\psi} \left( \begin{array}{c} a \\ D^* \end{array} \right) \Gamma \psi, \quad (a^2-DD^* \neq 0) \]

under Requirement 2. Now we put \(D = c \exp(i\theta)\) and apply a transformation

\(^*\) The reason of this nomenclature will be made clear in §5.
then the Lagrangian (I) is reduced to
\[ \mathcal{L} = -\bar{\Psi}' \left( \begin{array}{c} a \\ c \\ a' \end{array} \right) \mathcal{G}' \Psi', \quad (a^2 - c^2 > 0). \] (II)

Here $\Psi'$ is replaced by $\Psi$ for the sake of brevity.

The Lagrangian (II) is essentially equivalent to the Lagrangian (I), the most general one under restrictions 1 and 2. Therefore our discussion, limited to a q-number field, is hereafter based on the Lagrangian (II).

§ 3. Classifications of the Lagrangian

To classify the general Lagrangian (II), we apply a non-singular complex transformation\(^a\)
\[ \Psi = S \Psi' = \left( \begin{array}{cc} \rho & \sigma \\ \sigma^* & \rho^* \end{array} \right) \Psi', \quad \rho \rho^* - \sigma \sigma^* = 0. \] (3·1)

Then the Lagrangian (II) becomes
\[ \mathcal{L} = -\bar{\Psi}' S \mathcal{G}' \left( \begin{array}{c} a \\ c \\ a' \end{array} \right) \mathcal{G}' \Psi' = -\bar{\Psi}' \left( \begin{array}{cc} \mathcal{A} & \mathcal{D}^* \\ \mathcal{D} & \mathcal{A} \end{array} \right) \Psi', \] (3·2)

where the elements $\mathcal{A}$ and $\mathcal{D}$ are real and complex numbers respectively as given by (A·1) and (A·2). By choosing a transformation suitably, it is possible to make the Lagrangian simplest, that is,

i) diagonal form \[ \mathcal{A} = \text{real} \neq 0, \quad \mathcal{D} = 0, \]
or ii) off-diagonal form \[ \text{Re} \mathcal{D} = c \neq 0, \quad \mathcal{A} = \text{Im} \mathcal{D} = 0. \]

We give details of calculations in Appendix A and only list the results here. The primes of the wave-functions are dropped in transformed Lagrangians for the sake of simplicity.

i) diagonal form \((a^2 - c^2 > 0)\)

This is possible if and only if \(a^2 - c^2 > 0\), and the Lagrangian (II) becomes the Dirac-type one
\[ \mathcal{L} = -\mathcal{A} \bar{\Psi}' \mathcal{G} \Psi = -\mathcal{A} (\bar{\psi} \mathcal{G} \gamma^0 \psi + \bar{\psi}' \mathcal{G} \gamma^0 \psi') \] (3·3)
by a transformation

\(^a\) The transformation (2·31), is also a non-singular transformation of the same kind, so that we can get the same results as given below directly from the most general Lagrangian (I) by a single complex transformation, but calculations and results are much simpler and clearer when we deal with the Lagrangian (II). The same is true for the following transformations.
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\[ S = \frac{\sqrt{\lambda}}{2} \begin{pmatrix} s_x \exp (i\theta) & s_x \exp (-i\theta) \\ s_x \exp (i\theta) & s_x \exp (-i\theta) \end{pmatrix}, \quad (3.4) \]

where

\[ s_x = \sqrt{+1 \pm \sqrt{-1}} \]

with the simplified notations \( \sqrt{\pm} = \sqrt{|a \pm c|} \).

A sign of the coefficient \( \lambda \) is the same as that of the coefficient \( a \) in this case:

\[ \text{sign} \lambda = \text{sign} a, \quad (3.6) \]

therefore the Lagrangian (3·3) is nothing but the ordinary Dirac Lagrangian if \( \text{sign} \lambda > 0 \), i.e. in the case of \( \text{sign} a > 0 \).

ii) off-diagonal form \((a^2 - c^2 < 0)\)

This is possible if and only if \( a^2 - c^2 < 0 \), and the Lagrangian (II) becomes

\[ \mathcal{L} = -C\bar{\Psi} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Gamma \Psi = -C (\bar{\phi} \Gamma \psi + \bar{\phi} \Gamma \phi). \quad (3.7) \]

We call this a crossing-type Lagrangian. But in this case, \( \text{sign} \phi \) can be either the same or the opposite of \( \text{sign} c \) according as \((\rho + \rho^*)^2 + (\sigma - \sigma^*)^2\) is positive or negative. The corresponding transformation will be called ortho- or para-transformation respectively. They are

a) ortho-transformation (\( \text{sign} \phi = \text{sign} c \))

\[ S_0 = \frac{\sqrt{|\phi|}}{2} \begin{pmatrix} s_x \text{ch} \theta - is_x \text{sh} \theta & s_x \text{ch} \theta + is_x \text{sh} \theta \\ s_x \text{ch} \theta - is_x \text{sh} \theta & s_x \text{ch} \theta + is_x \text{sh} \theta \end{pmatrix}, \quad (3.8) \]

and

b) para-transformation (\( \text{sign} \phi = -\text{sign} c \))

\[ S_p = \frac{\sqrt{|\phi|}}{2} \begin{pmatrix} s_x \text{sh} \theta - is_x \text{ch} \theta & s_x \text{sh} \theta + is_x \text{ch} \theta \\ s_x \text{sh} \theta - is_x \text{ch} \theta & s_x \text{sh} \theta + is_x \text{ch} \theta \end{pmatrix}. \quad (3.9) \]

Another classification can also be given by understanding the \( \phi \) as a mixture of the two Majorana fields. Now we apply a non-singular transformation

\[ \Psi = M \Phi = \begin{pmatrix} \mu & \nu \\ \mu^* & \nu^* \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \quad \mu \nu^* - \mu^* \nu = 0 \quad (3.10) \]

to the Lagrangian (II). Here, the spinor fields \( \phi_1 \) and \( \phi_2 \) are the two Majorana fields; \( \phi_1 = \phi_1^c \) and \( \phi_2 = \phi_2^c \), and belong to the Lagrangian (2·29). Then the Lagrangian (II) becomes

\[ \mathcal{L} = -\bar{\Phi} M^a \begin{pmatrix} a & c \\ c & a \end{pmatrix} \Gamma M \Phi = -\bar{\Phi} \begin{pmatrix} f_1 & g \\ g & f_2 \end{pmatrix} \Gamma \Phi, \quad (3.11) \]

where the matrix elements \( f_1, f_2 \) and \( g \) are all real numbers, given by (B·1)
to (B·3). We can find a suitable transformation which makes the Lagrangian (3·11) a diagonal or an off-diagonal form. The details are given in Appendix B, and the results are as follows:

A) **diagonal form**

This is possible both for \(a^2-c^2\geq 0\), and the Lagrangian (II) becomes

\[
\mathcal{L} = -\bar{\phi} \begin{pmatrix} f_1 & 0 \\ 0 & f_2 \end{pmatrix} \Gamma \phi = -f_1 \bar{\phi}_1 \Gamma \phi_1 - f_2 \bar{\phi}_2 \Gamma \phi_2
\]

(3·12)

by a transformation \(M\) which makes the off-diagonal element \(g\) vanish:

\[
M_d = \begin{pmatrix} me^{i\phi} & \pm \frac{i}{\sqrt{m}} n (ae^{i\phi} + ce^{-i\phi}) \\ me^{-i\phi} & \mp \frac{i}{\sqrt{m}} n (ae^{-i\phi} + ce^{i\phi}) \end{pmatrix},
\]

(3·13)

whose non-singular condition is \(mn(a+c \cos 2\theta) \neq 0\), and use is made of the abbreviation

\[
\sqrt{=} = (a^2 + c^2 + 2ac \cos 2\theta)^{1/2} = [(a+c)^2 \cos^2 \theta + (a-c)^2 \sin^2 \theta]^{1/2}.
\]

(3·14)

Then the diagonal elements are given by

\[
f_1 = 2m^2 (a+c \cos 2\theta),
\]

\[
f_2 = 2n^2 (a+c \cos 2\theta) \frac{(a^2-c^2)}{(a^2+c^2+2ac \cos 2\theta)}.
\]

(3·15)

It is easy to get the following results:

A-i) If \(a^2-c^2>0\), the sign of \((a+c \cos 2\theta)\) is the same as that of \(a\), so that both \(f_1\) and \(f_2\) have the same sign as \(a\):

\[
f_1, f_2 \geq 0 \quad \text{if} \quad a \geq 0,
\]

(3·16)

A-ii) If \(a^2-c^2<0\), the sign of \((a+c \cos 2\theta)\) depends not only on the sign of \(c\) but on the magnitude of \(\theta\). In any case,

\[
f_1\text{ and } f_2 \text{ have opposite signs.}
\]

(3·17)

B) **off-diagonal form** \((a^2-c^2<0)\).

This is possible if and only if \(a^2-c^2<0\), and the Lagrangian (II) becomes

\[
\mathcal{L} = \mp 2mn(c + a \cos 2\theta) \bar{\phi} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Gamma \phi
\]

\[
= \mp 2mn(c + a \cos 2\theta) (\bar{\phi}_1 \Gamma \phi_1 + \bar{\phi}_2 \Gamma \phi_2),
\]

(3·18)

by a transformation

\[
M_{\phi\phi} = \begin{pmatrix} me^{i\phi} & \pm \frac{1}{\sqrt{m}} n (ae^{i\phi} + ce^{-i\phi}) \\ me^{-i\phi} & \mp \frac{1}{\sqrt{m}} n (ae^{-i\phi} + ce^{i\phi}) \end{pmatrix},
\]

(3·19)

However, this case is not so interesting and will be mentioned no more.
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Therefore, we find the following conclusions.
i) A spinor field, belonging to the Lagrangian (II) with \( a^2 - c^2 > 0 \), is to be considered as a mixture of the Dirac-type field and its charge conjugate field on the one hand, or as a mixture of the two definite or indefinite Majorana fields according as \( a > 0 \) or \( a < 0 \) on the other hand.

ii) A spinor field, belonging to the Lagrangian (II) with \( a^2 - c^2 < 0 \), is to be considered as a mixture of the crossing-type field and its charge conjugate field on the one hand, or as a mixture of a definite and an indefinite Majorana fields on the other hand.

§ 4. Gauge transformation and current density

As is easily seen, the Lagrangian (II) is not invariant against an ordinary gauge transformation, i.e. a phase transformation. However, we extend the definition of a gauge transformation to such a continuous complex one

\[ \Psi' = G \Psi = \left( \begin{array}{cc} \xi & \eta \\ \eta^* & \xi^* \end{array} \right) \Psi, \quad (4.1) \]

which makes the Lagrangian (II) invariant and continuously converges to the identical transformation by a limiting process. By this transformation, the Lagrangian (II) becomes the same form as (3.2) with the matrix elements (A.1) to (A.4) except that the \( \rho \) and \( \sigma \) are replaced by the \( \xi \) and \( \eta \). Hence a transformation is a gauge one if

\[ a(\xi \eta^* + \eta \xi^*) + c(\xi \eta + \xi^* \eta^*) = a, \]
\[ 2a(\xi \eta^* + \xi^* \eta) + c(\xi^2 + \xi^* \eta^2 + \eta^2 + \eta \eta^*) = 2c, \quad (4.2) \]
\[ 2a(\xi \eta^* - \xi^* \eta) + c(\xi^2 - \xi^* \eta^2 - \eta^2 + \eta \eta^*) = 0. \]

From these equations, we get a gauge transformation

\[ G = \text{sign} a \begin{pmatrix} ae^{i\rho} \sqrt{a^2 - c^2 \sin^2 \rho} & ic \sin \rho \sqrt{a^2 - c^2 \sin^2 \rho} \\ -ic \sin \rho \sqrt{a^2 - c^2 \sin^2 \rho} & ae^{-i\rho} \sqrt{a^2 - c^2 \sin^2 \rho} \end{pmatrix} \quad (4.3) \]

which depends on the coefficients \( a \) and \( c \), otherwise the transformation \( G \) can only be a multiple of an identical transformation. The argument of the square root must be positive, and actually it is always positive if \( a^2 - c^2 > 0 \). However, if \( a^2 - c^2 < 0 \), it is positive only if the angle \( \rho \) is smaller than a certain characteristic value, and furthermore it is unsatisfactory in the sense that the transformation (4.2) does not include the limiting case \( a = 0 \) and \( c \neq 0 \). Therefore it is more convenient to have a gauge transformation in respective cases. \(^*\)

\(^*\) The transformations \( G_+ \) and \( G_- \) can of course be derived directly in respective cases.
i) \(a^2 - c^2 > 0\)

Put \( \cos \theta = \text{sign}(a) \frac{a}{\sqrt{a^2 - c^2}} \sin \rho \), the transformation (4·2) becomes

\[
G_+ = \begin{pmatrix}
\cos \theta + i \text{sign}(a) \frac{a}{\sqrt{a^2 - c^2}} \sin \theta & i \text{sign}(a) \frac{c}{\sqrt{a^2 - c^2}} \sin \theta \\
-i \text{sign}(a) \frac{c}{\sqrt{a^2 - c^2}} \sin \theta & \cos \theta - i \text{sign}(a) \frac{a}{\sqrt{a^2 - c^2}} \sin \theta
\end{pmatrix}.
\] (4·2+)

ii) \(a^2 - c^2 < 0\)

Put \( \text{ch} \lambda = \text{sign}(a) \frac{a}{\sqrt{a^2 - c^2}} \sin \rho \), the transformation (4·2) becomes

\[
G_- = \begin{pmatrix}
\text{ch} \lambda + i \text{sign}(c) \frac{a}{\sqrt{c^2 - a^2}} \sin \lambda & i \text{sign}(c) \frac{c}{\sqrt{c^2 - a^2}} \sin \lambda \\
-i \text{sign}(c) \frac{c}{\sqrt{c^2 - a^2}} \sin \lambda & \text{ch} \lambda - i \text{sign}(c) \frac{a}{\sqrt{c^2 - a^2}} \sin \lambda
\end{pmatrix}.
\] (4·2−)

The transformation \(G_-\) covers the limiting case \(a=0\) and \(c=0\).

The sign factor of the transformation \(G, G_+\) or \(G_-\) is so chosen as it reduces respectively to

\[
\begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix} \text{ or } \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \text{ for vanishing } c,
\]

or

\[
\begin{pmatrix} \text{ch} \lambda & i \sin \lambda \\ -i \sin \lambda & \text{ch} \lambda \end{pmatrix} \text{ for vanishing } a.
\]

The interaction with an electromagnetic field is introduced by Utiyama's method of a generalized \(x\)-dependent gauge invariance. In our case, an infinitesimal gauge transformation of \(\mathcal{F}\) is

\[
\partial \mathcal{F} = i e \omega \varepsilon(x) \begin{pmatrix} a & c \\ -c & -a \end{pmatrix} \mathcal{F}.
\] (4·3)

Here

\[
\varepsilon(x) = \rho(x)/\epsilon, \quad \omega = 1/\epsilon, \quad \text{for } G(x), \quad (4·4)
\]

\[
\varepsilon(x) = \theta(x)/\epsilon, \quad \omega = \text{sign}(a)/\sqrt{a^2 - c^2}, \quad \text{for } G_+(x), \quad (4·4+)
\]

\[
\varepsilon(x) = \lambda(x)/\epsilon, \quad \omega = \text{sign}(c)/\sqrt{c^2 - a^2}, \quad \text{for } G_-(x), \quad (4·4−)
\]

and \(\rho(x), \theta(x)\) and \(\lambda(x)\) are infinitesimal functions. Since the gauge transformation group is one-parametric and commutative, an electromagnetic field induced transforms as

\[
\partial A_\mu = \partial \varepsilon(x)/\partial x_\mu. \quad (4·5)
\]

The field \(A_\mu\) is contained in a total Lagrangian only through a covariant
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Derivative

\[ V_{\mu} = \partial_{\mu} V - i e \omega \left( \begin{array}{c} a \\ -c \\ -a \end{array} \right) \mathcal{F} A_{\mu} . \]  

(4.6)

Thus a total Lagrangian is given by putting \( V_{\mu} \) in place of \( \partial_{\mu} V \) in the original free Lagrangian (II):

\[ \mathcal{L}_{\text{total}} = \mathcal{L}_{\text{free}} + \mathcal{L}_{\text{int}} \]  

(4.7)

where the \( \mathcal{L}_{\text{free}} \) consists of the free spinor Lagrangian (II) and the ordinary free electromagnetic Lagrangian. The interaction Lagrangian \( \mathcal{L}_{\text{int}} \) is given by

\[ \mathcal{L}_{\text{int}} = i e \omega (a^2 - c^2) \bar{\psi} \gamma_{\mu} \left( \begin{array}{c} 1 \\ 0 \\ -1 \end{array} \right) \mathcal{F} A_{\mu} = j_{\mu} (x) A_{\mu} (x) , \]  

(4.8)

and the quantity \( j_{\mu} (x) \) is a conserved current density, and is given by

\[
\begin{align*}
  j_{\mu} (x) &= i e \frac{a^2 - c^2}{a} \bar{\psi} \gamma_{\mu} \left( \begin{array}{c} 1 \\ 0 \\ -1 \end{array} \right) \mathcal{F} , & \text{for } G , \\
  j_{\mu} (x) &= i e \text{sign} (a) \sqrt{a^2 - c^2} \bar{\psi} \gamma_{\mu} \left( \begin{array}{c} 1 \\ 0 \\ -1 \end{array} \right) \mathcal{F} , & \text{for } G_{\pm} , \\
  j_{\mu} (x) &= i e \text{sign} (c) \sqrt{c^2 - a^2} \bar{\psi} \gamma_{\mu} \left( \begin{array}{c} 1 \\ 0 \\ -1 \end{array} \right) \mathcal{F} , & \text{for } G_{-} .
\end{align*}
\]

(4.9)

§ 5. Quantization

As is well known, the Majorana field, whose Lagrangian is \( \mathcal{S} = -i \bar{\phi} \gamma_{\mu} \phi \), is quantized in a definite metric space when \( f > 0 \). But we are here considering a quantization in the generalized Hilbert space, hence the coefficient \( f \) is not necessarily positive but negative. It is easily seen that the Majorana field with a negative \( f \) can actually be quantized in an indefinite metric space in a natural way. Thus we named a field with a positive \( f \) the definite Majorana field, and a field with a negative \( f \) the indefinite Majorana field in § 2.

As shown in § 3, a spinor field with \( a^2 - c^2 > 0 \) and \( a > 0 \) is a mixture of two definite Majorana fields, so that it is quantized in a definite metric space. Contrarily, a spinor field with \( a^2 - c^2 > 0 \) and \( a < 0 \) is a mixture of two indefinite Majorana fields, that is, it is purely indefinite and a usual indefinite metric quantization procedure is applied. On the other hand, a field with \( a^2 - c^2 < 0 \) is much more abnormal, because it is a mixture of a definite and an indefinite Majorana fields.

Now we deal with the Lagrangian (II); then the commutation relations (2.24) becomes

\[ \{ \psi (x) , \bar{\psi} (x') \} = \{ \psi^c (x) , \bar{\psi}^c (x') \} = -i \frac{a}{2 (a^2 - c^2)} S (x - x') , \]  

(5.1)
A canonical energy-momentum tensor derived by the usual procedure is

\[ T_{\mu\nu} = -\Phi \left( \frac{a}{c} \right) \gamma_{\mu\nu} \gamma^\nu \]  

and an energy-momentum four vector is

\[ P_{\nu} = \int T_{\mu\nu} d\sigma_{\mu} \]  

We expand wave functions in the usual way:

\[ \psi(x) = \frac{1}{\sqrt{V}} \sum_p \sum_r \{ a_r(p) u_r(p) e^{ipx} + b_r^+(p) v_r(p) e^{-ipx} \} \]

\[ \psi^c(x) = \frac{1}{\sqrt{V}} \sum_p \sum_r \{ a_r^+(p) v_r(p) e^{-ipx} + b_r(p) u_r(p) e^{ipx} \} \]

\[ \bar{\psi}(x) = \frac{1}{\sqrt{V}} \sum_p \sum_r \{ a_r^+(p) \bar{u}_r(p) e^{-ipx} + b_r(p) \bar{v}_r(p) e^{ipx} \} \]

\[ \bar{\psi}^c(x) = \frac{1}{\sqrt{V}} \sum_p \sum_r \{ a_r^+(p) \bar{v}_r(p) e^{ipx} + b_r^+(p) \bar{u}_r(p) e^{-ipx} \} \]

Here \( px = px - p_0 t \) and \( p_0 = (p^2 + m^2)^{1/2} \) and the \( c \)-number functions \( u_r \) and \( v_r \) are the usual Dirac eigenspinors. Putting these into (5.1), (5.2), (5.4) and (4.9), we get

\[ \{ a_r(p), a_{r'}^+(p') \} = \{ b_r(p), b_{r'}^+(p') \} = \frac{a}{2(a^2 - c^2)} \delta_{rr'} \delta_{pp'} \]

\[ \{ a_r(p), b_{r'}^+(p') \} = \{ a_{r'}(p), b_r(p') \} = \frac{c}{2(a^2 - c^2)} \delta_{rr'} \delta_{pp'} \]

Others = 0,

\[ P_{\nu} = \sum_r \sum_p p_{\nu} \left[ 2a_r a_{r'} + b_r b_{r'} - b_r b_{r'} + b_{r'} b_r \right] \]

\[ = \sum_r \sum_p p_{\nu} \left[ 2a_r a_{r'} + b_r b_{r'} + 2c(a_{r'} b_r + b_r a_{r'}) \right] - 1 \]

and the charge is expressed by

\[ Q = \int j_{\nu}(x) d\sigma_{\nu} = -ie\omega (a^2 - c^2) \sum_r \sum_p \left[ (a_{r'} a_r + b_r b_{r'}) - (b_{r'} b_r + a_r a_{r'}) \right] \]

\[ = -2ie\omega (a^2 - c^2) \sum_r \sum_p \left[ a_{r'} a_r - b_r^2 b_r \right] \]

Thus, the quantum operators \( a \) and \( b \) do not satisfy the Jordan-Wigner
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Commutation relations and cannot be understood as creation or annihilation operators of the field quantum.

Now in order to get an interpretation of creation and annihilation operators, we apply a non-singular complex transformation to the operators $a$ and $b$

\[
\alpha_r(p) = ka_r(p) + lb_r(p), \quad \beta_r(p) = ma_r(p) + nb_r(p),
\]

where the coefficients $k, l, m$ and $n$ are complex numbers. The inverse transformation is given by

\[
a_r(p) = \frac{[n\alpha_r(p) - l\beta_r(p)]}{D}, \quad b_r(p) = \frac{[-m\alpha_r(p) + k\beta_r(p)]}{D},
\]

where $D = kn - lm$.

Applying these transformations, we get

\[
\{\alpha_r(p), \alpha^*_r(p')\} = \frac{1}{2(a^2 - c^2)} [a(kk^* + ll^*) - c(kl^* + k^*l)] \delta_{rr}\delta_{pp'},
\]

\[
\{\beta_r(p), \beta^*_r(p')\} = \frac{1}{2(a^2 - c^2)} [a(mm^* + nn^*) - c(mn^* + m^*n)] \delta_{rr}\delta_{pp'},
\]

\[
\{\alpha_r(p), \beta^*_r(p')\} = \frac{1}{2(a^2 - c^2)} [a(kn^* + ln^*) - c(kn^* + lm^*)] \delta_{rr}\delta_{pp'},
\]

\[
\{\beta_r(p), \alpha^*_r(p')\} = \frac{1}{2(a^2 - c^2)} [a(k^*m + l^*n) - c(k^*n + l^*m)] \delta_{rr}\delta_{pp'}.
\]

An interpretation of creation and annihilation operators is allowed if these commutation relations reduce to the Jordan-Wigner type ones:

\[
\{\alpha_r(p), \alpha^*_r(p')\} = \epsilon_1 \delta_{rr}\delta_{pp'},
\]

\[
\{\beta_r(p), \beta^*_r(p')\} = \epsilon_2 \delta_{rr}\delta_{pp'},
\]

\[
\{\alpha_r(p), \beta^*_r(p')\} = \{\beta_r(p), \alpha^*_r(p')\} = \text{Others} = 0,
\]

where the factors $\epsilon_1$ and $\epsilon_2$ are sign factors $\pm 1$. Accordingly, we get

\[
P = \sum_r \sum_p \frac{P_r}{DD^*} [4(a^2 - c^2)(\epsilon_1 \alpha_r\alpha^*_r + \epsilon_2 \beta_r\beta^*_r) - 1].
\]

As will be shown in Appendix C, these sign factors $\epsilon_1$ and $\epsilon_2$ are fixed by the general properties

a) $\epsilon_1 \epsilon_2 = \text{sign}(a^2 - c^2)$,

b) $\epsilon_1 = \epsilon_2 = \text{sign} a$ if $a^2 - c^2 > 0$, (5.14)

c) $\epsilon_1$ and $\epsilon_2$ have opposite signs if $a^2 - c^2 < 0$. 


A general transformation is given by one of coefficients set (C·17), (C·19) or (C·21) with four parameters.

The charge $Q$ (5·8) becomes

$$Q = -2i\epsilon \omega (a^2 - c^2) / \mathcal{D}^* \sum \sum_p [ (m^* n^* - m^* m^*) \alpha_r^* \alpha_r^\dagger \beta_r + (l^* k^* - l^* k^* \beta_r^\dagger \alpha_r ) \beta_r ] ,$$

by the transformation (5·9). No coefficients of the terms in the square bracket vanish generally. However, if we further require that the coefficients of the terms $\alpha_r^\dagger \beta_r$ and $\beta_r^\dagger \alpha_r$ do vanish, we get more restricted two-parametric transformation under which the coefficients of the terms $\alpha_r^\dagger \alpha_r$ and $\beta_r^\dagger \beta_r$ have opposite signs with the same absolute value. This is possible, however, if and only if $a^2 - c^2 > 0$, as is shown at the end of Appendix C.

Furthermore, it is examined in Appendix D that if the transformation (5·9) can be understood as a transformation of a wave function as a whole, say $\psi' = \mu \phi' + \nu \phi'^\dagger$. This is also the case if and only if $a^2 - c^2 > 0$, and the transformation (5·9) is reduced to an one-parametric one which is a special case of the restricted two-parametric transformation mentioned above, and it transforms a field into the Dirac-type one. This is essentially equivalent to an inverse of the $S$-transformation (3·4).

In concluding this section, we give here only special examples of the operator transformation for the sake of simplicity and definiteness.$^*$

i) $a^2 - c^2 > 0$, \(\epsilon_1 = \epsilon_2 = \text{sign} \, a\),

$$a_r(p) = [ (\sqrt{+1} - \sqrt{-1}) \alpha_r(p) + (\sqrt{+1} - \sqrt{-1}) \beta_r(p) ] / 2\sqrt{2},$$
$$b_r(p) = [ (\sqrt{+1} - \sqrt{-1}) \alpha_r(p) + (\sqrt{+1} + \sqrt{-1}) \beta_r(p) ] / 2\sqrt{2} .$$

This is an $S$-transformation (3·4) and the charge $Q$ is expressed only by the terms $\alpha_r^\dagger \alpha_r$ and $\beta_r^\dagger \beta_r$ with the opposite signs.

ii-a) $a^2 - c^2 < 0$, \(\epsilon_1 = - \epsilon_2 = \text{sign} \, c\),

$$a_r(p) = [ \sqrt{+1} \alpha_r(p) + \epsilon_1 \sqrt{-1} \beta_r(p) ] / 2 ,$$
$$b_r(p) = [ \sqrt{+1} \alpha_r(p) - \epsilon_1 \sqrt{-1} \beta_r(p) ] / 2 .$$

ii-b) $a^2 - c^2 < 0$, \(- \epsilon_1 = \epsilon_2 = \text{sign} \, c\),

$$a_r(p) = [ \sqrt{-1} \alpha_r(p) - \epsilon_1 \sqrt{+1} \beta_r(p) ] / 2 ,$$
$$b_r(p) = [ \sqrt{-1} \alpha_r(p) + \epsilon_1 \sqrt{+1} \beta_r(p) ] / 2 .$$

By these transformations, the charge $Q$ is expressed only by the terms $\alpha_r^\dagger \beta_r$ and $\beta_r^\dagger \alpha_r$ with the same sign.

$^*$) See Appendix C for the notation.
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§ 6. Concluding remarks

A spinor field satisfying the Dirac equation is generally analyzed in the light of the Lagrangian method in the generalized Hilbert space. In the case \( a^2 - c^2 > 0 \) and \( a > 0 \), the analysis and the results are essentially equivalent to those of Takahashi, Kamefuchi and Tanaka in connection with the Jauch field, and the field is quantized in the ordinary Hilbert space. In the other case \( a^2 - c^2 > 0 \) and \( a < 0 \) or \( a^2 - c^2 < 0 \), it is necessary to generalize the Hilbert space to an indefinite space. The treatment is rather trivial in the case \( a^2 - c^2 > 0 \) and \( a < 0 \). But in the case \( a^2 - c^2 < 0 \), the particle aspect of the field can only be introduced by a new treatment.

Now we see a typical case of \( a^2 - c^2 < 0 \), say, a crossing-type Lagrangian corresponding to \( a = 0 \) and \( c = 1/2 \). The Lagrangian is

\[
\mathcal{L} = -\frac{1}{2} \left( \overline{\psi} \Gamma \psi^c + \overline{\psi}^c \Gamma \psi \right). \tag{6.1}
\]

The commutation relations (5.1) and (5.2) are

\[
\{ \psi(x), \overline{\psi}(x') \} = \{ \psi^c(x), \overline{\psi}^c(x') \} = 0, \\
\{ \psi(x), \overline{\psi}^c(x') \} = \{ \psi^c(x), \overline{\psi}(x') \} = -iS(x - x'), \tag{6.2}
\]

Others = 0.

According to the expression (5.17), the wave function is expanded as

\[
\psi(x) = \frac{1}{\sqrt{2V}} \sum_p \left\{ [\alpha_r(p) + \beta_r(p)] u_r(p)e^{ipx} + [\alpha^*_r(p) - \beta^*_r(p)] v_r(p)e^{-ipx} \right\}, \tag{6.3}
\]

and the operators \( \alpha_r(p) \), etc., satisfy the Jordan-Wigner type commutation relations

\[
\{ \alpha_r(p), \alpha^*_r(p') \} = \delta_{r, r'} \delta_{pp'}, \\
\{ \beta_r(p), \beta^*_r(p') \} = -\delta_{r, r'} \delta_{pp'}, \tag{6.4}
\]

Others = 0.

Therefore, the operators \( \alpha_r(p) \), \( \beta_r(p) \), etc., can be interpreted as a creation or an annihilation operator. Though the \( \alpha \)-quantum is treated in an ordinary positive metric space, the \( \beta \)-quantum should be treated in an indefinite metric space. The energy-momentum \( P \) is diagonal in this representation, i.e. it is expressed only by the occupation number terms \( \alpha_r \alpha_r \) and \( \beta_r \beta_r \) with the opposite signs. However, the charge \( Q \) is not the case, so that the charged \( \psi \) cannot be described suitably by this treatment. There is no such defect for an uncharged field.

A massless neutrino field is usually described by a field \( \phi(x) \) which satisfies equation...
but the parity non-conservation allows for a neutrino a wave equation

\[
[\gamma_5 \partial_\mu + \kappa (1 \pm \gamma_5) / 2] \psi (x) = 0 ,
\]

(6.6)

where \( \kappa \) is an arbitrary constant with the dimension of an inverse length. The Lagrangian which gives this equation is a crossing-type one, and is treated as above. The details of this case will appear in this journal.\(^5\)

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Appendix A

The transformation \( S \)

By the transformation \( S \) (3.1), the Lagrangian (II) is transformed to the Lagrangian (3.2) with

\[
\bar{\mathcal{L}} = a (\rho \rho^* + \sigma \sigma^*) + c (\rho \sigma + \rho^* \sigma^*)
= a (r^2 + r'^2 + s^2 + s'^2) + 2c (rs - r's') ,
\]

(\( A \cdot 1 \))

\[
\mathcal{D} = 2a \rho \sigma^* + c (\rho^2 + \sigma^2) ,
\]

(\( A \cdot 2 \))

\[
\text{Re } \mathcal{D} = a (\rho \sigma^* + \rho^* \sigma) + c (\rho^2 + \rho^*2 + \sigma^2 + \sigma^*2)
= 2a (rs + r's') + c (r^2 - r'^2 + s^2 - s'^2) ,
\]

(\( A \cdot 3 \))

\[
i \text{Im } \mathcal{D} = a (\rho \sigma^* - \rho^* \sigma) + c (\rho^2 - \rho^*2 - \sigma^2 + \sigma^*2)
= 2i [a (rs - r's') + c (rr' - ss')] .
\]

(\( A \cdot 4 \))

Thus, the diagonal term \( \bar{\mathcal{L}} \) is real but the off-diagonal term \( \mathcal{D} \) is complex, and the second lines of these expressions are given by putting

\[
\rho = r + ir' , \quad \sigma = s + is' .
\]

(\( A \cdot 5 \))

The non-singular condition of the transformation is

\[
\rho \rho^* - \sigma \sigma^* = r^2 + r'^2 - s^2 - s'^2 \neq 0 .
\]

(\( A \cdot 6 \))

i) diagonal form

We require \( \mathcal{D} = 0 \). From a condition \( \text{Im } \mathcal{D} = 0 \), we get

\[
s' = r' (as + cr) / (ar + cs) .
\]

(\( A \cdot 7 \))

Putting this into a condition \( \text{Re } \mathcal{D} = 0 \), we get


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\[ [2ars + c(r^3 + s^3)] [1 + r^3 (a^2 - c^2)/(ar + cs)^2] = 0. \] (A·8)

However, the latter term cannot vanish because of the non-singular condition

\[ \rho \rho^* - \sigma \sigma^* = (r^3 - s^3) [1 + r^3 (a^2 - c^2)/(ar + cs)^2] \neq 0, \] (A·9)

which is given by putting (A·7) into (A·6). Hence a condition \( \text{Re } \mathcal{D} = 0 \) is reduced to

\[ 2ars + c(r^3 + s^3) = 0. \] (A·10)

Putting

\[ \alpha = -(as + cr)/(ar + cs), \] (A·11)

we get

\[ s = \alpha r, \quad s' = -\alpha r', \quad \text{i.e.} \quad \sigma = \alpha \rho^*, \] (A·12)

from (A·7) and (A·10), and \( \alpha \) satisfies

\[ c\alpha^2 + 2a\alpha + c = 0. \] (A·13)

In order that the quantities \( r \) and \( s \) are both real, the quantity \( \alpha \) should be real, and so the discriminant \( a^2 - c^2 \) of Eq. (A·13) must be non-negative. Therefore we can make the Lagrangian (3·2) diagonal if and only if

\[ a^2 - c^2 > 0, \] (A·14)

because the case \( a^2 - c^2 = 0 \) is the Majorana theory and is rejected here.

Putting (A·12) into (A·1), we get

\[ \mathcal{A} = (ace^3 + 2c\alpha + a) \rho \rho^* = a[(\alpha + c/a)^3 + (a^2 - c^2)/a^2] \rho \rho^*. \] (A·15)

Since \( \rho \rho^* \) and the square bracket term are both positive, we get

\[ \text{sign } \mathcal{A} = \text{sign } a. \] (A·16)

The explicit solution (3·4) follows from (A·12), (A·13) and (A·15) by considering four possible cases \( a + c > a - c > 0 \), etc., separately.

ii) off-diagonal form

We require \( \text{Im } \mathcal{D} = 0 \) and \( \mathcal{A} = 0 \). Considering just as above, we get

\[ s = \beta r, \quad r' = -\beta s', \] (A·12')

where the quantity \( \beta \) satisfies

\[ a\beta^2 + 2c\beta + a = 0. \] (A·13')

We can make the Lagrangian (3·2) off-diagonal if and only if the discriminant of Eq. (A·13') is non-negative:

\[ c^2 - a^2 > 0, \] (A·14')

except for the Majorana case \( c^2 - a^2 = 0 \).

Putting (A·12') into (A·3), we get
\[ C = \text{Re} \mathcal{U} = (c \beta^2 + 2a \beta + c) \left( r^2 - s^2 \right) \]

\[ = c \left[ \left( \beta + a/c \right)^2 + \left( c^2 - a^2 \right)/c^2 \right] \left( r^2 - s^2 \right). \]  

(A·15')

Here the square bracket term is positive but \( r^2 - s^2 \) can be either positive or negative,*) so that sign \( C \) can be either the same or the opposite of sign \( c \) according as \( r^2 - s^2 \) is positive or negative.

The explicit solutions (3·8) and (3·9) follow from (A·12'), (A·13') and (A·15').

Appendix B

The transformation \( M \)

By the transformation \( M \), (3·10), the Lagrangian (II) is transformed to the Lagrangian (3·11) with

\[ f_1 = 2a \mu \mu^* + c \left( \mu^2 + \mu^* \right) = 2m^2 (a + c \cos 2\theta), \]

(B·1)

\[ f_2 = 2a \nu \nu^* + c \left( \nu^2 + \nu^* \right) = 2n^2 (a + c \cos 2\varphi), \]

(B·2)

\[ g = a \left( \mu \nu^* + \mu^* \nu \right) + c \left( \mu \nu + \mu^* \nu^* \right) = 2mn \left[ a \cos (\theta - \varphi) + c \cos (\theta + \varphi) \right]. \]

(B·3)

These are obviously real numbers, and the last terms are given by putting

\[ \mu = m \exp(i\theta), \quad \nu = n \exp(i\varphi). \]

(B·4)

The non-singular condition of the transformation is

\[ \mu \nu^* - \mu^* \nu = 2i mn \sin (\theta - \varphi) \neq 0. \]

(B·5)

A) diagonal form

A condition of vanishing the off-diagonal term \( g \) is

\[ a \cos (\theta - \varphi) + c \cos (\theta + \varphi) = 0. \]

(B·6)

From this, we can get sin \( \varphi \) and cos \( \varphi \):

\[ \sin \varphi = \pm \left( a + c \right) \cos \theta / \sqrt{a^2 + c^2 + 2ac \cos 2\theta}, \]

(B·7)

\[ \cos \varphi = \mp \left( a - c \right) \sin \theta / \sqrt{a^2 + c^2 + 2ac \cos 2\theta}. \]

Therefore

\[ \nu = ne^{i\varphi} = \pm in \left( ae^{i\theta} + ce^{-i\theta} / \sqrt{a^2 + c^2 + 2ac \cos 2\theta}. \]

(B·8)

The non-singular condition becomes

\[ \mu \nu^* - \mu^* \nu = \pm 2i mn (a + c \cos 2\theta) / \sqrt{a^2 + c^2 + 2ac \cos 2\theta} \neq 0, \]

(B·9)

i.e.

\[ mn (a + c \cos 2\theta) \neq 0. \]

(B·10)

*) \( r^2 - s^2 \) can also be zero, but the transformation becomes singular in this case and is rejected.
Thus the transformation is given by the matrix $M_d$ (3.13) in the text.

The diagonal terms $f_1$ and $f_2$ of (3.15) are easily derived from (B.1), (B.2) and (B.7).

B) off-diagonal form

Conditions of vanishing the diagonal terms $f_1$ and $f_2$ are

$$a + c \cos 2\theta = 0, \quad a + c \cos 2\varphi = 0.$$  \hspace{1cm} (B.11)

These are satisfied if and only if $|c| > |a|$, i.e. $c^2 - a^2 > 0$. If this is the case, Eqs. (B.11) give $\cos 2\theta = \cos 2\varphi$, i.e. $\theta \pm \varphi = N\pi$. However, $\theta - \varphi = N\pi$ does not satisfy the non-singular condition (B.5). Thus

$$\theta + \varphi = N\pi \quad (N: 0 \text{ or } \pm \text{integer}),$$ \hspace{1cm} (B.12)

and the results in the text follows.

**Appendix C**

The transformation of the operators $a_r(p)$ and $b_r(p)$ into the Jordan-Wigner operators

The relations to fix a transformation which transforms the operator $a_r(p)$ and $b_r(p)$ into the Jordan-Wigner operators are given by comparing the commutation relations (5.11) and (5.12). They are

$$a(kk* + ll*) - c(kl* + k*l) = 2(a^2 - c^2) \epsilon_1,$$ \hspace{1cm} (C.1)

$$a(mm* + nn*) - c(mn* + m*n) = 2(a^2 - c^2) \epsilon_1,$$ \hspace{1cm} (C.2)

$$a(km* + ln*) - c(kn* + lm*) = 0.$$ \hspace{1cm} (C.3)

Here the quantities $\epsilon_1$ and $\epsilon_2$ are sign factors $\pm 1$.

From Eq. (C.3), we get

$$n = m(ak* - cl*) / (ck* - al*).$$ \hspace{1cm} (C.4)

Putting this into Eq. (C.2) and using Eq. (C.1), we get

$$2(a^2 - c^2) \epsilon_1 = 2(a^2 - c^2) \epsilon_2 mm* / (ck - al) (ck* - al*).$$ \hspace{1cm} (C.5)

Since $mm* / (ck - al) (ck* - al*) > 0$, we get

$$\epsilon_1 \epsilon_2 = \text{sign} (a^2 - c^2).$$ \hspace{1cm} (C.6)

Furthermore, Eq. (C.1) can be transferred to

$$a[(k - lc/a) (k* - l*c/a) + ll* (a^2 - c^2) / a^2)] = 2(a^2 - c^2) \epsilon_1.$$ \hspace{1cm} (C.7)

Here the square bracket term is positive if $a^2 - c^2 > 0$, so that

$$\epsilon_1 = \epsilon_2 = \text{sign} a$$ \hspace{1cm} if $a^2 - c^2 > 0$$ \hspace{1cm} (C.8)

considering the relation (C.6).
On the other hand, we have no such restrictions if \( a^2 - c^2 < 0 \) and we can only say

\[ c) \quad \epsilon_1 \text{ and } \epsilon_2 \text{ have opposite signs if } a^2 - c^2 < 0. \]

From Eqs. (C·5) and (C·4), we get

\[
m = (ck^* - al^*) e^{\theta} / \sqrt{|a^2 - c^2|}, \quad n = (ak^* - cl^*) e^{\theta} / \sqrt{|a^2 - c^2|}. \tag{C·9}
\]

Hence the non-singular condition of the transformation is

\[
D = kn - lm = 2(a^2 - c^2) \epsilon_1 e^{\theta} / \sqrt{|a^2 - c^2|} \neq 0. \tag{C·10}
\]

Putting these into the left-hand side of Eq. (C·2), we get the left-hand side of Eq. (C·1) multiplied by \( \text{sign}(a^2 - c^2) \) which is equal to \( 2(a^2 - c^2) \epsilon_1 \) considering the relation (C·6). Hence, Eq. (C·2) is automatically satisfied by the coefficient \( m \) and \( n \) given by \( k \) and \( l \) satisfying Eq. (C·1). Putting

\[
k = \kappa + \lambda, \quad l = \kappa - \lambda, \tag{C·11}
\]

Eq. (C·1) becomes

\[
(a - c) \kappa \kappa^* + (a + c) \lambda \lambda^* = (a^2 - c^2) \epsilon_1. \tag{C·12}
\]

This is satisfied by the following \( \kappa \) and \( \lambda \).

\[ i) \quad a^2 - c^2 > 0, \quad \kappa = \sqrt{\pm} \cos \theta e^{i \rho}, \quad \lambda = \sqrt{\mp} \sin \theta e^{i \sigma}, \tag{C·13}
\]

\[ ii-a) \quad a^2 - c^2 < 0, \quad \epsilon_1 = \text{sign } c, \quad \kappa = \sqrt{\pm} \cosh \theta e^{i \rho}, \quad \lambda = \sqrt{\mp} \sinh \theta e^{i \sigma}, \tag{C·14}
\]

\[ ii-b) \quad a^2 - c^2 < 0, \quad \epsilon_1 = - \text{sign } c, \quad \kappa = \sqrt{\pm} \sinh \theta e^{i \rho}, \quad \lambda = \sqrt{\mp} \cosh \theta e^{i \sigma}, \tag{C·15}
\]

where use is made of the simplified notations

\[
\sqrt{+} = \sqrt{|a + c|}, \quad \sqrt{-} = \sqrt{|a - c|}. \tag{C·16}
\]

For the sake of symmetry, change \( \theta \rightarrow 2\theta \), \( \rho \rightarrow \rho + \theta \), \( \sigma \rightarrow \sigma + \theta \), and we get the transformations and the inverse transformations as follows:

\[ i) \quad a^2 - c^2 > 0, \quad \epsilon_1 = \epsilon_2 = \text{sign } a, \]

\[
e^{i \theta} \left( \begin{array}{cc}
\sqrt{+} \cos \theta e^{i \rho} + \sqrt{-} \sin \theta e^{i \sigma} & \sqrt{+} \cos \theta e^{i \rho} - \sqrt{-} \sin \theta e^{i \sigma} \\
-\epsilon_1 [\sqrt{-} \cos \theta e^{-i \rho} - \sqrt{+} \sin \theta e^{-i \sigma}] & \epsilon_1 [\sqrt{-} \cos \theta e^{-i \rho} + \sqrt{+} \sin \theta e^{-i \sigma}]
\end{array} \right). \tag{C·17}
\]

Since \( D = 2 \epsilon_1 \sqrt{a^2 - c^2} e^{2i \theta} \), the inverse transformation is

\[
e^{-i \theta} \left( \begin{array}{cc}
\sqrt{+}^{-1} \cos \theta e^{-i \rho} + \sqrt{-}^{-1} \sin \theta e^{-i \sigma} & -\epsilon_1 [\sqrt{-}^{-1} \cos \theta e^{i \rho} - \sqrt{+}^{-1} \sin \theta e^{i \sigma}]
\end{array} \right)
- \frac{1}{2} \left( \begin{array}{cc}
\sqrt{+}^{-1} \sin \theta e^{-i \rho} - \sqrt{-}^{-1} \cos \theta e^{-i \sigma}
\end{array} \right); \tag{C·18}
\]
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\[ \begin{align*}
\text{ii-a)} \quad & a^2 - c^2 < 0, \quad \epsilon_1 = - \epsilon_2 = \text{sign } c, \\
& \epsilon^\theta \left( \begin{array}{c} \sqrt{+} \text{ ch } \theta e^{i\varphi} + \sqrt{-} \text{ sh } \theta e^{i\varphi} \\ \epsilon_1 [\sqrt{-} \text{ ch } \theta e^{-i\varphi} + \sqrt{+} \text{ sh } \theta e^{-i\varphi}] \end{array} \right) \\
& \quad \times \epsilon^\varphi \left( \begin{array}{c} \sqrt{+} \text{ ch } \theta e^{i\varphi} - \sqrt{-} \text{ sh } \theta e^{i\varphi} \\ - \epsilon_1 [\sqrt{-} \text{ ch } \theta e^{-i\varphi} - \sqrt{+} \text{ sh } \theta e^{-i\varphi}] \end{array} \right). 
\end{align*} \]

Since \( \Delta = -2 \epsilon_1 \sqrt{c^2 - a^2} e^{2i\theta} \), the inverse transformation is

\[ \begin{align*}
\epsilon^{-i\varphi} \left( \begin{array}{c} \sqrt{-1} \text{ ch } \theta e^{-i\varphi} - \sqrt{-1} \text{ sh } \theta e^{-i\varphi} \\ \epsilon_1 [\sqrt{-1} \text{ ch } \theta e^{i\varphi} - \sqrt{+1} \text{ sh } \theta e^{i\varphi}] \end{array} \right) \\
2 \left( \begin{array}{c} \sqrt{-1} \text{ ch } \theta e^{-i\varphi} + \sqrt{-1} \text{ sh } \theta e^{-i\varphi} \\ - \epsilon_1 [\sqrt{-1} \text{ ch } \theta e^{i\varphi} + \sqrt{+1} \text{ sh } \theta e^{i\varphi}] \end{array} \right). 
\end{align*} \]

\( (C \cdot 19) \)

\[ \begin{align*}
\text{ii-b)} \quad & a^2 - c^2 < 0, \quad - \epsilon_1 = \epsilon_2 = \text{sign } c, \\
& \epsilon^\varphi \left( \begin{array}{c} \sqrt{-} \text{ sh } \theta e^{i\varphi} + \sqrt{+} \text{ ch } \theta e^{i\varphi} \\ - \epsilon_1 [\sqrt{-} \text{ sh } \theta e^{-i\varphi} + \sqrt{+} \text{ ch } \theta e^{-i\varphi}] \end{array} \right) \\
& \quad \times \epsilon^{-i\varphi} \left( \begin{array}{c} \sqrt{+} \text{ sh } \theta e^{-i\varphi} - \sqrt{-} \text{ ch } \theta e^{-i\varphi} \\ \epsilon_1 [\sqrt{-} \text{ sh } \theta e^{i\varphi} - \sqrt{+} \text{ ch } \theta e^{i\varphi}] \end{array} \right). 
\end{align*} \]

Since \( \Delta = -2 \epsilon_1 \sqrt{c^2 - a^2} e^{2i\theta} \), the inverse transformation is

\[ \begin{align*}
e^{-i\varphi} \left( \begin{array}{c} \sqrt{-1} \text{ sh } \theta e^{-i\varphi} - \sqrt{-1} \text{ ch } \theta e^{-i\varphi} \\ - \epsilon_1 [\sqrt{-1} \text{ sh } \theta e^{i\varphi} - \sqrt{+1} \text{ ch } \theta e^{i\varphi}] \end{array} \right) \\
- \frac{1}{2} \left( \begin{array}{c} \sqrt{-1} \text{ sh } \theta e^{-i\varphi} + \sqrt{-1} \text{ ch } \theta e^{-i\varphi} \\ \epsilon_1 [\sqrt{-1} \text{ sh } \theta e^{i\varphi} + \sqrt{+1} \text{ ch } \theta e^{i\varphi}] \end{array} \right). 
\end{align*} \]

\( (C \cdot 21) \)

The transformation is more restricted if we further require that the charge \( (5 \cdot 14) \) is expressed only by the terms \( \alpha_i \alpha_r \) and \( \beta_i \beta_r \). The conditions of vanishing the coefficients of the terms \( \alpha_i \beta_r \) and \( \beta_i \alpha_r \) give

\[ \frac{m}{l^*} = \frac{n}{k^*} = \eta, \quad \text{i.e.,} \quad m = \eta l^*, \quad n = \eta k^*. \]

\( (C \cdot 23) \)

Putting these into Eqs. \( (C \cdot 1) \) and \( (C \cdot 2) \), we get

\[ \text{[left side of Eq. (C \cdot 2)] = [left side of Eq. (C \cdot 1)].} \]

\( (C \cdot 24) \)

Hence the sign factor \( \epsilon_1 \) must be equal to the sign factor \( \epsilon_2 \), and this is the case if and only if \( a^2 - c^2 > 0 \). In this case, we have

\[ \eta \eta^* = 1, \quad \text{i.e.,} \quad \eta = \exp(i\tau). \]

\( (C \cdot 25) \)

Therefore, putting the relations \( (C \cdot 23) \) and \( (C \cdot 25) \) into the coefficients of the term \( \alpha_i \alpha_r \) of the charge \( Q \), we get a relation

\[ \text{[coefficient of } \alpha_i \alpha_r] = - \text{[coefficient of } \beta_i \beta_r]. \]

\( (C \cdot 26) \)

The transformation coefficients \( (C \cdot 17) \) are reduced to two-parametric ones.
Appendix D

Is it possible to understand the transformation of operators $a_r(p)$ and $b_r(p)$ into the Jordan-Wigner operators as a wave function transformation $\psi' = \mu \psi + \nu \psi^c$?

Now let operators $a_r(p)$ and $b_r(p)$ be expansion coefficients of wave functions $\varphi$ and $\varphi^c$, say the expression $(5 \cdot 5)$, and satisfy the commutation relations $(5 \cdot 6)$. On the other hand, we imagine that the Jordan-Wigner operators $\alpha_r(p)$ and $\beta_r(p)$ are expansion coefficients of a wave function $\varphi'$:

$$\varphi'(x) = \frac{1}{\sqrt{\psi}} \sum_p \{ \alpha_r(p) u_r(p) e^{ipx} + \beta_r(p) v_r(p) e^{-ipx} \}.$$  

We examine if a transformation $(5 \cdot 9)$ of the operators $a_r(p)$ and $b_r(p)$ into the Jordan-Wigner operators $\alpha_r(p)$ and $\beta_r(p)$ can be understood as a transformation of a wave function as a whole; $\varphi' = \mu \psi + \nu \psi^c$.

In this case, the operator transformation is written as

$$\alpha_r = \mu a_r + \nu b_r, \quad \beta_r = \nu^* a_r + \mu^* b_r.$$  

This is found to be the case

$$m = l^k, \quad n = k^*,$$

in the general transformation $(5 \cdot 9)$, which is nothing but the case $\eta=1$ in the transformation $(C \cdot 23)$, and is the case if and only if $a^2 - c^2 > 0$. The charge $Q$ is expressed only by the terms $\alpha_r \alpha_r$ and $\beta_r \beta_r$, with the opposite signs.

The transformation is one-parametric and is given by

$$\mu = \pm \frac{1}{\sqrt{2}} i^{N+M+1} \left[ \sqrt{+} + (-1)^{N+M+1/2} \sqrt{-} \right] e^{ip},$$

$$\nu = \pm \frac{1}{\sqrt{2}} i^{N+M+1} \left[ \sqrt{+} - (-1)^{N+M+1/2} \sqrt{-} \right] e^{ip}.$$  

Here the quantities $N$ and $M$ are integers and $\delta$ is 0 or 1 according as the integer $M$ is even or odd respectively.

We can easily show that this transformation is essentially the same as an inverse of the $S$-transformation $(3 \cdot 4)$ which gives a general field as a linear combination of the Dirac-type field and its charge conjugate one.

References
2) J. Schwinger, Phys. Rev. 82 (1951), 914; 91 (1953), 713.