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The Use of the Proper Time in Quantum Electrodynamics I.

Yôichirô NAMBU

*Department of Physics, University of City Osaka**

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I. Introduction

The space-time approach to quantum electrodynamics, as has been developed by Feynman,[1] seems to offer a very attractive and useful idea to this domain of physics. His ingenious method is indeed attractive, not only because of its intuitive procedure which enables one to picture to oneself the complicated interactions of elementary particles, its ease and relativistic correctness with which one can calculate the necessary matrix elements or transition probabilities, but also because of its way of thinking which seems somewhat strange at first look and resists our minds that are accustomed to causal laws. According to the new standpoint, one looks upon the world in its four-dimensional entirety. A phenomenon that will come into play in this theatre is now laid out beforehand in full detail from immemorial past to ultimate future and one investigates the whole of it at glance. The time itself loses sense as the indicator of the development of phenomena; there are particles which flow down as well as up the stream of time; the eventual creation and annihilation of pairs that may occur now and then, is no creation nor annihilation, but only a change of directions of moving particles, from past to future, or from future to past; a virtual pair, which, according to the ordinary view, is foredoomed to exist only for a limited interval of time, may also be regarded as a single particle that is circulating round a closed orbit in the four-dimensional theatre; a real particle is then a particle whose orbit is not closed but reaches to infinity. . .

In such a view, a state with prescribed number of particles including real as well as virtual does not exactly correspond to a four-dimensional state in the ordinary sense, that is, a state represented by the wave function satisfying the time dependent Schroedinger equation. But the former is rather a part of the latter in which any number of virtual particles may be allowed to occur. To obtain an idea of the actual state we shall have to sum over all possibilities as to the number of virtual particles.

The interpretation of the four-dimensional state in the present sense becomes also somewhat different from the conventional one as giving the transition pro-

*) Now staying temporarily at Tokyo University.

bability or amplitude from a given state A to a final state F in the three-dimensional space. We can rather ask for the relative probability that a four-dimensional state A-I-F with prescribed real as well as virtual (intermediate) particles be realized in nature. This will be zero unless the arbitrary chosen A-I-F is not such as is an actually possible transition under the Schroedinger equation.

The above-mentioned view of the entire space-time behavior of nature *sub specie aeternitatis*, however, might not appeal to a reason which is liable to think in the language of differential equations and pursue the development of things along a certain parameter. In fact we find it hard to regard the world line of a particle as a mere status of that particle, but are unconciously following the motion of an imaginary mass point along the world line. Thus, in Feynman's theory where the ordinary time loses its rôle as the indicator of the development of the world, it would still be convenient to introduce some parameter with which the four-dimensional world is going to shape itself. How this is possible to a certain extent we shall see in what follows.

2. Formal introduction of the proper time

Let us consider a wave function obeying the ordinary Schroedinger equation

$$i\frac{\partial}{\partial t}\psi(t,x)=H(x)\,\psi(t,x). \tag{1}$$

The scalar product (ψ,φ) of two wave functions ψ and φ at a given time carries the meaning of the probability amplitude for finding a state characterized by ψ when we know that the system is in the state φ. This interpretation is based upon the mathematical fact that the length of the wave function vector is constant in time according to (1). If we go over to the four-dimensional standpoint and regard the behavior of ψ in both t and x for a finite interval of time as characterizing the state $\Psi(t,x)$, we shall naturally have to define the norm of a wave function by

$$(\Psi,\Psi)=\int_{t_0}^{t_0+T}\int_{-\infty}^{\infty}(\psi(tx),\psi(tx))dx\,dt=T\int(\psi(x),\psi(x))\,dx, \tag{2}$$

which is a multiple of the ordinary norm of the state ψ. Thus we have only to alter the normalization of the wave function for the transition from ψ to Ψ, though this means an infinite factor when the time interval is extended indefinitely. The probability amplitude (ψ,φ) mentioned above is then in the new standpoint merely a probability amplitude density for a cross section of time from which the full amplitude is derived by integration with respect to t:

$$(\Psi,\Phi)=\int(\psi,\phi)\,dt. \tag{3}$$

Eq. (3) allows the following interpretation. Suppose we take for Ψ and Φ two stationary states for a system, then (3) will be zero unless Ψ and Φ represents the same state. But if some perturbation is introduced in the system, the system will no longer remain in the original (three-dimensional) state φ, but the space-time behavior of Φ will be expressed by decomposing it with respect to unperturbed eigenfunctions,

$$\Phi=\sum_k a_k(t)\,\Phi_k(t), \qquad a_k(t_0)=\delta_{k0}. \tag{4}$$

The probability that we find the system after an infinite lapse of time in the state k is given by

$$\lim_{t\to\infty}|a_k(t)|^2=\lim_{t\to\infty}|(\psi_k,\Phi)|^2. \tag{5}$$

If we displace the time scale and shift the initial point to $-\infty$, we obtain the probability for finding a state Ψ_k or ψ_k (referred to unperturbed coordinates) irrespective of time when we know the system was in the state φ_0 and the perturbation has been, say adiabatically, switched on, by

$$|a_k(\infty)|^2=|\int_{-\infty}^{\infty}(\Psi_k(t),\Phi(t))\,dt|^2=|(\Psi_k,\Phi)|^2, \tag{6}$$

for we may suppose that $a_k(t)$ has reached its stationary value for any finite t. In this way the four-dimensional scalar product acquires a physical meaning.

Now let us investigate the problem from a different point of view. The Schroedinger equation (1) implies, when regarded four-dimensionally, a sort of supplementary condition imposed on Ψ:

$$(i\partial/\partial t-H)\Psi=0 \tag{7}$$

since we no longer look upon t as a parameter along which Ψ develops itself. Consequently a Ψ which corresponds to the real world must be of the form

$$\Psi_{\mathrm{real}}=\delta\left(i\frac{\partial}{\partial t}-H\right)\Psi_0. \tag{8}$$

If we introduce here a redundant variable τ and assume an equation of the type

$$i\frac{\partial}{\partial\tau}\Psi=\left(i\frac{\partial}{\partial t}-H\right), \tag{9}$$

and an accompanying eigenvalue problem

$$\lambda\Psi=\left(i\frac{\partial}{\partial t}-H\right)\Psi, \tag{9'}$$

(8) becomes

$$\Psi_{\mathrm{real}}=\delta(\lambda)\,\Psi_0=\frac{1}{2\pi}\int_{-\infty}^{\infty}\Psi(\tau)\,d\tau. \tag{10}$$

The four-dimensional view has a static character in that a state is defined once for all t and after that a condition is invoked in order that it correspond to any reality. But now that τ is introduced and (9) has regained the aspect of an ordinary Schroedinger equation, the new variable must play a similar rôle as that the ordinary time played in (1). As τ goes on, a wave packet localized over certain four-dimensional volume will move to and fro and change its shape gradually. Then we may think of τ as something like the proper time of the particle represented by the wave packet. If we directed a camera on to that particle with the shutter open for an infinitely long time, we should obtain a vague strip of world line as the locus of the particle, which would correspond to the real wave function Ψ_{real}. Eq. (10) tells us just this situation in the mathematical language.

Next we shall consider the transition probability. If a perturbation term is inserted, (9) becomes

$$i\frac{\partial}{\partial t}\Psi=\left(i\frac{\partial}{\partial t}-H_0-H_1\right)\Psi, \tag{11}$$

while the free particle wave functions to which we refer the initial and final state satisfy the equation

$$\left(i\frac{\partial}{\partial t}-H_0\right)\Psi_A=\left(i\frac{\partial}{\partial t}-H_0\right)\Psi_F=0. \tag{12}$$

Let us write $i\partial/\partial t-H_0\equiv L$, and perform the transformation

$$\Psi=\exp[-iL\tau]\,\Psi_1.\quad H_1'=\exp[iL\tau]\,H_1\exp[-iL\tau], \tag{13}$$

(11) then goes over into

$$i\frac{\partial}{\partial t}\Psi_1=-H_1'\Psi_1. \tag{14}$$

Starting at $\tau=\tau_0$ from a free state Ψ_A, the transition amplitude at $\tau=\tau$ to a free state Ψ_F will be expressed by

$$P(AF)=\delta(L)(F|U(\tau\tau_0)|A),\quad \Psi(\tau)=U(\tau\tau_0)\,\Psi(\tau_0), \tag{15}$$

since H_1' would in general bring a real particle into unreal one for which $L\Psi\neq 0$. Eq. (15) admits a twofold interpretation: a) formally, $P(AF)$ is given from $(\Psi_F, \Psi(\tau))$ by taking the zero-frequency component of $\Psi(\tau)$:

$$P(AF)=\frac{1}{2\pi}\int_{-\infty}^{\infty}(\Psi_F, \Psi(\tau))\,d\tau\,; \tag{16}$$

b) physically, it is the accumulated amplitude for the transition starting from the state A and arriving at F after an infinite lapse of time:

$$P(AF)=\int_{-\infty}^{\infty}(\Psi_F, \Psi(\tau))\,d\tau=\int_{-\infty}^{\infty}(F|U(\tau,-\infty)|A)\,d\tau. \tag{17}$$

(16) and (17) differ only by a normalization factor.

Thus we see that the formal introduction of a redundant parameter and its identification with the proper time carries a bit of mathematical convenience as well as physical plausibility. In order to convince ourselves further on this point, we shall next recall Fock's theory and go ahead on his line.

3. Theory of Fock and its extension

Fock[2] once introduced the concept of the proper time in the Dirac electron in parallel with the classical theory and proved its correspondence to the ordinary proper time. It will be briefly recapitulated below. The classical Lagrangian for an electron interacting with the electromagnetic field is given by

$$L_0 = -mc\dot{x}_\mu^2 - mc^2/2 - (e/c)(\dot{x}_\mu A_\mu), \quad \dot{x}_\mu = dx_\mu/d\tau, \tag{18}$$

and the equation of motion follows from the variational principle

$$\delta S = 0, \quad S = \int^\tau L_0 d\tau, \tag{19}$$

together with the supplementary condition

$$\dot{x}_\mu^2 = \text{const.} = -c^2. \tag{20}$$

Eliminating τ from (19) and (20) we get the ordinary action function $S(x_\mu)$ for the system. The Hamilton-Jacobi partial differential equation which follows from above is

$$\frac{\partial S}{\partial \tau} + \frac{1}{2m}\left[\left(\text{grad}\, S + \frac{e}{c}A\right)^2 - \frac{1}{c}\left(\frac{\partial S}{\partial t} - e\Phi\right)^2\right] + m^2c^2 = 0, \tag{21}$$

with the condition

$$\partial S/\partial \tau = 0. \tag{21'}$$

Now turning to quantum theory, the Dirac electron obeys the wave equation

$$(\gamma_\mu D_\mu + \varkappa)\psi = 0, \quad D_\mu = \partial/\partial x_\mu - e/\hbar c \cdot A_\mu. \tag{22}$$

Such a ψ may be expressed as

$$\psi = (\gamma_\mu D_\mu - \varkappa)\Psi, \tag{23}$$

where Ψ in turn shall obey the second order differential equation

$$(\gamma D + \varkappa)(\gamma D - \varkappa)\Psi = \left[\left(\Box - \frac{ei}{\hbar c}A\right)^2 - \varkappa^2 + \frac{e}{\hbar c}\sigma_{\mu\nu}F_{\mu\nu}\right]\Psi \equiv \Lambda\Psi = 0. \tag{24}$$

Fock proposed to introduce a function F which satisfies instead of (24) an equation closely analogous to (21):

$$i\hbar\frac{\partial}{\partial \tau}F = \frac{\hbar^2}{2mc}\Lambda F, \tag{25}$$

and express Ψ as a suitable integral of F over τ:

$$\Psi=\int_C F d\tau, \tag{26}$$

the contour being taken in such a way that

$$\frac{\partial}{\partial\tau}\Psi=\int_C\frac{\partial F}{\partial\tau}d\tau=F|_C=0, \tag{27}$$

which corresponds to the condition (21′).

When we neglect temporarily the electromagnetic interaction, the electron becomes free and (25) is solved by

$$F(\tau)=\exp\left(-\frac{i\hbar\tau}{2mc}\Lambda\right)F(0). \tag{28}$$

Given an initial distribution of the wave packet over space-time, it will change with τ by the relation (28). But only those stationary states with zero eigenvalue can ever correspond to a real world:

$$\Lambda F=\lambda F=0. \tag{29}$$

In classical language, a preassigned pattern of streamlines representing the motion of an aggregate of electrons will change according to the dynamical law (21). Only the stationary stream is the true state of the world where individual electrons follow invariant paths.

Now let us determine the behavior of the wave packet which starts from a delta function $F(0)=\delta(x)$ at $\tau=0$. The answer is easily given by the Fourier representation

$$\begin{aligned}F(\tau,x)&=\left(\frac{1}{2\pi}\right)^4\int e^{i\lambda\tau}e^{ik_\mu x_\mu}\,\delta(\lambda+(k^2+\varkappa^2)/2\varkappa)\,dk_\mu d\lambda\\&=\frac{\varkappa}{\pi}\int\Delta^{(1)}(x^2,\varkappa^2+2\varkappa\lambda)\,e^{i\lambda\tau}d\tau\equiv\varkappa\Delta_\varkappa(\tau,x^2)\\&=-\frac{i}{4\pi^2}\frac{\varkappa^2}{\tau^2}\exp\left(\frac{\varkappa}{2\tau}x^2 i+\frac{\varkappa}{2}\tau i\right),\qquad(\tau>0).\end{aligned} \tag{30}$$

$F(\tau,x)$ is the probability amplitude for finding the particle at x when we know that it was at the origin a time τ ago. According to the previous argument the real observed amplitude will be obtained when we sum $F(\tau,x)$ for all $\tau>0$, which results in

$$\begin{aligned}\int_0^\infty F(\tau,x)\,d\tau&=\frac{2\varkappa}{(2\pi)^3}\int_0^\infty\cdot\delta_+(k^2+\varkappa^2)\,e^{ik_\mu x_\mu}dk_\mu\\&=\frac{2\varkappa}{i}\left(\bar{\Delta}(x)+\frac{i}{2}\Delta^{(1)}(x)\right)=\varkappa\Delta_F.\end{aligned} \tag{31}$$

Here $\bar{\Delta}$, $\Delta^{(1)}$, and Δ_F bear the meaning as defined in Schwinger,[3] Feynman,[1] and Dyson[4] :

$$\bar{\Delta}(x)=\frac{1}{(2\pi)^4}\int\frac{1}{k^2+\kappa^2}\,e^{ik_\mu x_\mu}\,dk_\mu,$$

$$\Delta^{(1)}(x)=\frac{1}{(2\pi)^3}\int\delta(k^2+\kappa^2)\,e^{ik_\mu x_\mu}\,dk_\mu,$$

$$\Delta_F(x)=-2i\bar{\Delta}(x)+\Delta^{(1)}(x)=\frac{2}{(2\pi)^3}\int\delta_+(k^2+\kappa^2)\,e^{ik_\mu x_\mu}\,dk_\mu. \tag{32}$$

Thus we see that we shall be able to arrive at Δ_F, the fundamental quantity in positron theory, if we take $F(\tau, x)$ for the (five-dimensional) commutation relation between quantized wave functions satisfying (25). The proper time τ is nothing but what has been a mere parameter in the integral representation of the Δ-functions. The integration with respect to τ followed above, not from $-\infty$ to $+\infty$ but only for positive τ, is a departure from the standpoint expounded before, and corresponds to the fact that four-dimensional outgoing waves divergent from a source into past and future are exclusively considered in Feynman's theory.

Now the starting equation (25) seems somewhat artificial and impairing the simplicity of the original Dirac equation. In fact one would be tempted to introduce the proper time as a fifth coordinate in a linearized form. Thus one may put for instance

$$\gamma_5\frac{\partial}{\partial\tau}\psi=\left(\gamma_\mu\frac{\partial}{\partial x_\mu}+\kappa\right)\psi,\quad \gamma_5\frac{\partial}{\partial\tau}\psi^\dagger=\left(\gamma_\mu\frac{\partial}{\partial x_\mu}-\kappa\right)\psi^\dagger, \tag{33}$$

which, by iteration, yields

$$\left(\frac{\partial^2}{\partial\tau^2}+\Box^2-\kappa^2\right)\psi=\left(\frac{\partial^2}{\partial\tau^2}+\Box^2-\kappa^2\right)\psi^\dagger=0. \tag{34}$$

A solution of (24) can be written as

$$\psi=\left(\gamma_5\frac{\partial}{\partial\tau}+L^\dagger\right)\Phi,\quad (\Box_\sigma^2-\kappa^2)\,\Phi=0,$$

$$L=\gamma_\mu\frac{\partial}{\partial x_\mu}+\kappa,\quad L^\dagger=\gamma_\mu\frac{\partial}{\partial x_\mu}-\kappa,\quad \sigma=1,\ldots\ldots5,\ x_5\equiv\tau.. \tag{35}$$

For the commutation relation we adopt the expression

$$\{\psi,\psi^\dagger\}=\left(\gamma_5\frac{\partial}{\partial\tau}+L^\dagger\right)f(\tau,x),$$

$$f(\tau,x)=-\frac{2i}{(2\pi)^4}\int_c e^{ik_\sigma x_\sigma}\,\delta(k_\sigma^2+\kappa^2)\,dk_\sigma, \tag{36}$$

where C means the path of integration for k_5: $+\infty i \to 0 \to +\infty$. First, when integrated over $\tau = x_5$ from 0 to ∞, (36) gives

$$\int_0^\infty f(\tau, x)\, d\tau = -\frac{2i}{(2\pi)^4}\int_C e^{ik_\mu x_\mu}\,\delta(k_\sigma^2 + \varkappa^2)\left[\pi\delta(k_5) - \frac{1}{ik_5}\right] dk_\sigma$$

$$= -\frac{2i}{(2\pi)^4}\int e^{ik_\mu x_\mu}\left[\frac{\pi}{2}\,\delta(k_\mu^2+\varkappa^2) + \frac{1}{2i(k_\mu^2+\varkappa^2)}\right] dk_\mu = -\frac{i}{2}\,\Delta_F(x), \quad (37)$$

and (36) reduces to

$$\int_0^\infty \{\psi(\tau),\ \psi^{\dagger\prime}(0)\}\, d\tau = -\frac{i}{2} L^\dagger \Delta_F(x), \quad (38)$$

the term with $\partial/\partial\tau$ being put to zero on integration. The initial value of f for $\tau = 0$ is calculated in a similar way, and yields the result

$$\left(\frac{\partial}{\partial\tau}\gamma_5 - L^\dagger\right) f\bigg|_{\tau=0} = \frac{-i}{(2\pi)^4}\int\left(i\gamma_5 - \frac{L^\dagger}{(-k_\mu^2 - \varkappa^2)^{1/2}}\right) e^{ik_\mu x_\mu}\, dk_\mu. \quad (39)$$

But the factor in the bracket becomes

$$i\gamma_5 - \frac{L^\dagger}{(-k_\mu^2 - \varkappa^2)^{1/2}} = i\gamma_5(1+\varepsilon), \quad \varepsilon = \gamma_5 L^\dagger/(-k_\mu^2 - \varkappa^2)^{1/2}, \quad (40)$$

$$\varepsilon^2 = 1, \quad \varepsilon = \pm 1.$$

Thus, (39) is not of the form const. $\delta(x)$, as it should be. This is because we are selecting only those waves which are outgoing from the source point. Neglecting ε, or taking its average, we get the reasonable result

$$<\{\psi(x),\ \gamma_5\psi^\dagger(x')\}>_{\tau=\tau'} \sim \delta(x). \quad (41)$$

In this point the linearized form does not prove to be much preferable to the original Fock equation. Though it may be convenient when we try to make use of the interaction representation, we shall follow hereafter Fock's procedure which seems most natural after all.

4. Problem of vacuum polarization

Now we shall turn to the interpretation of the various terms of the S matrix. The starting point is that the probability amplitude for an electron going from x to x' in a time τ is assumed to be given by

$$(\Psi^*,\ \psi(x,\tau)) = \left(\gamma_\mu\frac{\partial}{\partial x_\mu} - \varkappa\right)\Delta(\tau, x-x') \equiv S(\tau, x-x'). \quad (42)$$

If, for example, we consider the self-energy of the electron, we have to do with the process: an electron and a photon start simultaneously at x and afterwards meet again at x', thereby giving rise to the matrix element

$$-\frac{e^2}{\hbar c}\gamma_\mu S(\tau, x-q')\gamma_\mu D_F(x-x'). \tag{43}$$

Integrating over τ we get the usual self-energy element. We could also introduce another proper time for the photon at least formally, modifying $D(x-x')$ to $D(\tau, x-x')$. But then the radiation field no more remains real.

The problem of the vacuum polarization is a more interesting subject. Here we shall confine ourselves to the external polarization only. An electron, starting from x, suffers a scattering by the field at x' (time τ), then comes back to the origin (time $\tau+\tau'$) and there produces a polarization current δj_μ. This will be given by

$$\delta j_\mu = i\frac{e}{\hbar c}Tr[\gamma_\mu S(\tau', x-x')A_\nu\gamma_\nu S(\tau, x'-x)]. \tag{44}$$

Integration with respect to τ and τ' leads to the usual exppession. If we consider the possibility that an electron can be scattered by the field over and over again before return, then we are dealing with an electron moving under continuous influence of the field. Such an electron will be described by the wave function satisfying

$$\lambda\Psi_\lambda = \Lambda\Psi_\lambda. \tag{45}$$

Its contribution to the induced current is

$$i\int_0^\infty(\Psi^*(x)\,e^{-i\lambda\tau}\gamma_\mu\Psi_\lambda(x))\,d\tau = 2\pi i\delta_+(\lambda)\,\Psi_\lambda^*(x)\,\gamma_\mu(\gamma_\nu D_\nu - \varkappa)\,\Psi_\lambda(x). \tag{46}$$

Summing over all stationary states (and adjusting the normalization factor), we get the induced current expression

$$\delta j_\mu = i\sum_\lambda \delta_f(\lambda)\,\Psi_\lambda^*\gamma_\mu(\gamma_\nu D_\nu - \varkappa)\,\Psi_\lambda. \tag{47}$$

Letting the external field vanish, (47) reduces simply to

$$\delta j_\mu^0 = i\,\mathrm{Tr}\left[\gamma_\mu\left(\gamma_\lambda\frac{\partial}{\partial x_\lambda} - \varkappa\right)\frac{1}{(2\pi)^4}\int\delta_+(k^2+\varkappa^2)\,e^{ik_\mu x_\mu}\,dk_\mu\right]_{x=0} = \frac{i}{2}\,\mathrm{Tr}\,[\gamma_\mu S_F(0)], \tag{48}$$

which may be regarded as zero by virtue of symmetry. If, however, (47) is expanded in powers of the coupling constant, an expression of the form

$$\delta j_\mu = c_1 A_\mu + c_2\Box^2 A_\mu + \cdots\cdots \tag{49}$$

will be obtained. The first and second term are divergent, corresponding to the self-energy of photon and the renormalization of the external charge respectively. We shall show, however, that there is a different method of approach to determine the form of δj_μ.

Fock resorted to a kind of the W.K.B. method to solve the proper time wave equation (25), setting

$$F = e^{\frac{i}{\hbar}S}f, \tag{50}$$

where S is the classical action function satisfying (21). This method is applied to obtain the Riemann function R for the Dirac electron, which is expressed as

$$R=\oint F d\tau, \tag{51}$$

the contour encircling the origin $\tau=0$ in the complex domain. F can be rigorously solved when the electron is free or at least the external field is constant. In the former case, in particular,

$$S=+\frac{m}{2\tau}x^2-\frac{1}{2}mc^2\tau, \qquad f=-\frac{m}{8\pi^2\hbar c}\frac{1}{\tau^2},$$

$$R=-\frac{m}{8\pi^2\hbar c}\oint e^{\frac{im}{2\hbar\tau}x^2-\frac{imc^2}{2\hbar}\tau}\frac{d\tau}{\tau^2}=-\frac{m}{4\pi\hbar\sqrt{-x^2}}J_1\left(\frac{mc}{\hbar}\sqrt{-x^2}\right). \tag{52}$$

On the other hand, Schwinger's $\bar{\Delta}$-function has the integral representation

$$\bar{\Delta}(x)=\frac{1}{8\pi^2}\int_{-\infty}^{\infty}e^{-i\alpha x^2+i\frac{x^2}{4\alpha}}d\alpha=\frac{x}{16\pi^2}\int_{-\infty}^{\infty}e^{-\frac{ix^2}{2\tau}x+\frac{ix\tau}{2}}\frac{d\tau}{\tau^2}. \tag{53}$$

When $-x^2$ is positive, we may take a contour integral going round the upper half plane at infinity and deviating slightly below the real axis near the origin; when $-x^2$ is negative, we go round the lower half plane at infinity. Then the former contour shrinks to a circle around zero, giving just (43), and the latter integral vanishes. Thus

$$\bar{\Delta}(x)=-\frac{1}{2}R(x)\gamma(-x^2), \tag{54}$$

where

$$\gamma(\lambda)=\begin{cases}1, & \lambda>0\\ 0, & \lambda<0\end{cases}$$

From this result we see that the function $\Delta_F^{(e)}$ in the presence of the external field will be obtained if we choose in (42) a straight integration path from 0 to ∞ instead of the circle around zero. When there is a constant magnetic and electric field H and E parallel to the z axis, $F=\exp(iS/\hbar)f$ is given by

$$S=S_0-\frac{1}{2}mc^2\tau+\frac{eE}{4c}[(z-z')^2-c^2(t-t')^2]\operatorname{cth}\frac{eE\tau}{2mc}$$

$$+\frac{eH}{4c}[(x-x')^2+(y-y')^2]\operatorname{ctg}\frac{eH\tau}{2mc},$$

$$S_0=\frac{1}{2}eE(z+z')(t-t')+\frac{eH}{2c}(x'y-y'x), \tag{55}$$

$$f=\frac{m}{8\pi^2\hbar c}\frac{eH}{2mc}\frac{eE}{2mc}f_0\Big/\sin\frac{eH\tau}{2mc}\operatorname{sh}\frac{eE\tau}{2mc},$$

$$f_0=\exp\left[\frac{ie}{2mc}\sigma_z H\tau+\frac{e}{2mc}\alpha_z E\tau\right],$$

and

$$-\frac{1}{2}\Delta_F^{(e)}(x,x')=\int_0^\infty F(x,x')\,d\tau. \tag{56}$$

The polarization current is then

$$\delta j_\mu=i\,Tr\left[\gamma_\mu\left(\gamma_\lambda\frac{\partial}{\partial x_\lambda}-\varkappa-\frac{ei}{\hbar c}\gamma_\lambda A_\lambda\right)\Delta_F^{(e)}(x,x')\right]_{x=x'}, \tag{57}$$

where

$$A_x=-\frac{1}{2}Hy,\quad A_y=\frac{1}{2}Hx,\quad A_z=0,\quad A_4=Ezi. \tag{58}$$

Let us consider the problem of the gauge invariance. When A_μ is replaced by $A_\mu+\partial\chi/\partial x_\mu$, S acquires an additional term $(e/c)(\chi-\chi')$ so that $\Delta_F^{(e)}$ is multiplied by a factor $\exp[i(\chi(x)-\chi(x')/\hbar]$. But the current (48) turns out to suffer no modification:

$$\delta j_\mu'=e^{i(\chi-\chi')/\hbar}\,\delta j_\mu|_{x=x'}=\delta j_\mu. \tag{59}$$

The gauge invariance is thus guaranteed. The charge conservation law also holds since

$$\left(\gamma\Box+\varkappa-\frac{ei}{\hbar c}A\gamma\right)S_F^{(e)}=S_F^{(e)}\left(-\gamma\Box'+\varkappa-\frac{ie}{\hbar c}A'\gamma\right)=-\delta(x-x'). \tag{60}$$

These relations are a consequence of the equation (25) for F and the initial condition: $F(x)=\delta(x)$ at $\tau=0$.

On the other hand the usual perturbation formula (40) gives in general non-gauge invariant, divergent results when evaluated numerically. In our expression (56) for the particular constant field, however, δj_μ turns out to vanish if we regard expressions of the type $(x_\mu-x_\mu')\Delta_F^{(e)}(x,x')|_{x=x'}$ to be zero. This may be approved because here appears only one singular function and not a product of two or more, so that it is a consistent condition compatible with other requirements. It is also shown in the appendix that no essential difficulties arise in case of an arbitrary constant field. The self-energy of the photon is then zero at least in the order e^2, for the first term in the expansion (40) vanishes according to the above result.

Appendix. Solution for an arbitrary constant external field.[5)]

We take the constant external field $F_{\lambda\mu}=-F_{\mu\lambda}$, and the vector potential for it:

$$A_\mu=\frac{1}{2}(x_\lambda-x_{0\lambda})F_{\lambda\mu}, \tag{A1}$$

where $x_{0\lambda}$ are arbitrary constants corresponding to an initial condition in the subsequent calculation. The classical equation of motion is then

$$\frac{d}{d\tau}\dot{x}_\mu = -\frac{e}{mc}F_{\mu\lambda}\dot{x}_\lambda$$

or

$$\frac{d}{d\tau}\dot{x} = -\frac{e}{mc}\boldsymbol{F}\dot{x}, \qquad \boldsymbol{F}=(F_{\mu\lambda}). \tag{A2}$$

The Hamilton-Jacobi equation is, on the other hand, given by

$$\frac{\partial S}{\partial\tau}+\frac{1}{2m}\left[\left(\frac{\partial S}{\partial x_\mu}+\frac{e}{2c}(x_\lambda-x_{0\lambda})F_{\lambda\mu}\right)^2+m^2c^2\right]=0. \tag{A3}$$

Instead of solving (A3) directly, we shall first handle the equation of motion (A2) which can easily be integrated because $\boldsymbol{F}$ is a constant matrix. Thus,

$$\dot{\boldsymbol{x}}=\exp\left[-\frac{e}{mc}\boldsymbol{F}\tau\right]\dot{\boldsymbol{x}}_0\equiv\boldsymbol{S}\dot{\boldsymbol{x}}_0, \tag{A4}$$

and further

$$(\boldsymbol{x}-\boldsymbol{x}_0)=\left[(1-\boldsymbol{S}(\tau))\Big/\frac{e}{mc}\boldsymbol{F}\right]\dot{\boldsymbol{x}}_0,$$

$$\therefore\quad \dot{\boldsymbol{x}}_0=\left[\frac{e\boldsymbol{F}}{mc}\Big/(1-\boldsymbol{S}(\tau))\right](\boldsymbol{x}-\boldsymbol{x}_0), \tag{A5}$$

$$\dot{\boldsymbol{x}}=\boldsymbol{S}\dot{\boldsymbol{x}}_0=\left[\frac{e\boldsymbol{F}}{mc}\boldsymbol{S}\Big/(1-\boldsymbol{S})\right](\boldsymbol{x}-\boldsymbol{x}_0).$$

The conjugate momenta p_μ and the Hamiltonian $H=(p\dot{x})-L$ become

$$\begin{aligned}\boldsymbol{p}&=m\dot{\boldsymbol{x}}-(e/c)\boldsymbol{A}=m\dot{\boldsymbol{x}}+(e/2c)\boldsymbol{F}(\boldsymbol{x}-\boldsymbol{x}_0)\\&=\left[\frac{e\boldsymbol{F}}{c}\boldsymbol{S}\Big/(1-\boldsymbol{S})+\frac{e\boldsymbol{F}}{2c}\right](\boldsymbol{x}-\boldsymbol{x}_0)=\frac{e\boldsymbol{F}}{2c}(1+\boldsymbol{S})/(1-\boldsymbol{S})\cdot(\boldsymbol{x}-\boldsymbol{x}_0)\\&=\frac{e}{2c}\boldsymbol{F}\operatorname{cth}\left(\frac{e}{2mc}\boldsymbol{F}\tau\right)\cdot(\boldsymbol{x}-\boldsymbol{x}_0),\end{aligned} \tag{A6}$$

$$\begin{aligned}H&=\frac{1}{2}mc^2+\frac{1}{2m}\left(\boldsymbol{p}+\frac{e}{c}\boldsymbol{A}\right)^2=\frac{1}{2}mc^2+\frac{1}{2}m\dot{\boldsymbol{x}}^2\\&=\frac{1}{2}mc^2+\frac{1}{2}mc^2\cdot\frac{e^2}{m^2c^2}\left(\frac{\boldsymbol{FS}}{1-\boldsymbol{S}}\boldsymbol{x}-\boldsymbol{x}_0,\ \frac{\boldsymbol{FS}}{1-\boldsymbol{S}}\boldsymbol{x}-\boldsymbol{x}_0\right)\\&=\frac{1}{2}mc^2+\frac{e^2}{2m}\left(\boldsymbol{x}-\boldsymbol{x}_0,\ \frac{-\boldsymbol{F}^2\boldsymbol{S}}{(1-\boldsymbol{S})^2}\boldsymbol{x}-\boldsymbol{x}_0\right),\end{aligned} \tag{A7}$$

using the relation for the transposed:

$$\tilde{\boldsymbol{S}}=\boldsymbol{S}^{-1}, \qquad \tilde{\boldsymbol{F}}=-\boldsymbol{F}. \tag{A8}$$

The action function S, which satisfies $\delta S = p\delta x - H\delta\tau$, is now obtained by putting

$$S = -\frac{1}{2} mc^2\tau + \frac{e}{4c}(\boldsymbol{x} - \boldsymbol{x}_0, \boldsymbol{G}(\boldsymbol{x} - \boldsymbol{x}_0)), \tag{A9}$$

$$\boldsymbol{G} = \boldsymbol{F} \operatorname{cth}\left(\frac{e}{2mc}\boldsymbol{F}\tau\right).$$

The remaining function f is determined by the equation

$$2m\frac{df}{d\tau} + \left(\Box^2 \boldsymbol{S} + \frac{ei}{2c}\sigma_{\mu\nu}F_{\mu\nu}\right)f = 0 \tag{A10}$$

From (A9),

$$\Box^2 \boldsymbol{S} = \frac{e}{2c} Tr\, \boldsymbol{G},$$

so that

$$f = \exp\left[-\frac{1}{2} Tr\left\{\ln \operatorname{sh}\left(\frac{e}{2mc}\boldsymbol{F}\tau\right) + \frac{ei}{2mc}(\sigma\boldsymbol{F})\tau\right\}\right] f_0. \tag{A11}$$

In these expressions, functions of the matrix $\boldsymbol{F}$ bear symbolical meaning. When it happens that det $|\boldsymbol{F}| = 0$ a d consequently no inverse can be defined directly, we should go back to the beginning for the correct interpretation.

Now that the functions $F = \exp(iS/\hbar)f$ and A_μ are both even in the coordinates $x - x'$, we see that the argument proposed in Section 4 for the discussion of the induced current holds also in this general case.

References

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