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Reciprocal Symmetries of the Dual-Resonance Propagators

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It is explicitly shown that integral expressions of the dual-resonance propagators defined on two-dimensional (one-"space" and one-"time") media exhibit a detailed symmetry when the "space" axis and the "time" axis are interchanged in some sense with each other. For example, the dual-resonance propagator in momentum space proves to get a parametric integral representation of the form quite similar to the representation of the position-space propagator through this interchange. In particular, this reciprocal symmetry is shown to have an interesting connection with the recently known critical dimension of space for which no ghosts appear.

It is likely that the symmetry has its origin in a graph-theoretical duality inherent in two-dimensional planar media, but its prototype may also be found in the well-known properties of the conventional Feynman propagators.

§ 1. Introduction

As is widely known, it is possible to represent both the conventional Feynman propagator in position space and that in momentum space by similar parametric integrals. These expressions exhibit a reciprocity which exists between them. In fact we can write

$$\Delta_F(x-x') = \int_0^\infty d\alpha e^{-i\alpha m_0^2/2} \int \mathcal{D}^4 x(t) \exp\left\{-(i/2)\int_0^\alpha dt(\partial x(t)/\partial t)^2\right\}$$
$$= -(i/4\pi^2)\int_0^\infty d\alpha\alpha^{-2}\exp\{-(i/2)[m_0^2\alpha+(x-x')^2/\alpha]\} \qquad (1\cdot1)^{*)}$$

for the former $(x(\alpha)=x', x(0)=x)$, while the momentum-space propagator or the Fourier transform of $(1\cdot1)$ takes the form

$$(p^2-m_0^2)^{-1} = -(i/2)\int_0^\infty d\alpha \exp\{-(i/2)(m_0^2-p^2)\alpha\} \qquad (1\cdot2)$$

up to a constant factor. Let us tentatively introduce dual positions y and y' by

$$\alpha p = y-y'. \qquad (1\cdot3)$$

Then $(1\cdot2)$ becomes

$$(p^2-m_0^2)^{-1}|_{(1\cdot3)} = -(i/2)\int_0^\infty d\alpha \exp\{-(i/2)[m_0^2\alpha-(y-y')^2/\alpha]\}. \qquad (1\cdot4a)$$

*) We use the metric $g^{00}=-g^{11}=\dots=1$. Then $(x, p)=\sum g^{\mu\mu}x^\mu p^\mu$, whereas we write p^2 for (p, p).

It is evident that, in (1·1) and (1·4a), the exponential factor in one integrand takes a shape similar to that of the other.

Choice is, however, not unique and if we instead introduce y and y' by

$$p = y - y' \tag{1·3'}$$

and change the integration parameter (proper time) α to β by

$$\beta = 1/\alpha\,, \tag{1·5}$$

then we have

$$(p^2 - m_0^2)^{-1}|_{(1\cdot 3')} = -(i/2)\int_0^\infty d\beta\beta^{-2}\exp\{-(i/2)[m_0^2/\beta - (y-y')^2/\beta]\}\,. \tag{1·4b}$$

This time the factor β^{-2} appears as a factor symmetrically corresponding to α^{-2} in (1·1) [up to the present we have considered the space-time to be four-dimensional].

The symmetry which we have just described is rather trivial but has a deep connection with the reciprocal symmetries which lie among the field theoretical quantities variously in position and momentum spaces. (Refer, e.g., to Ref. 1)).*)

In a series of papers[2)~5)] we have brought about propagators which the dual-resonance models require. These new propagators just correspond to two-dimensional generalizations of (1·1) or (1·2). As an extension of the position-space propagator (1·1), we have defined (in Refs. 3) and 5))

$$\begin{aligned}\Delta_F[x_2(s), x_1(s)] &= \int_0^\infty d\alpha e^{-m_0^2\alpha/2}\int \mathcal{D}^8 x(s,t)\exp\{-\tfrac{1}{2}D[x(s,t); D_0^\alpha]\} \\ &= \int_0^\infty d\alpha e^{-m_0^2\alpha/2} g_x[\alpha]\exp\{-\tfrac{1}{2}D[x^h(s,t); D_0^\alpha]\}\,,\end{aligned} \tag{1·6}$$

where $D[x(s,t); D_0^\alpha]$ is the Dirichlet integral of $x(s,t)$ over the domain D_0^α:

$$D[x; D_0^\alpha] = \int_{D_0^\alpha} dsdt[(\partial x/\partial s)^2 + (\partial x/\partial t)^2]$$

and $x^h(s,t)$ denotes the harmonic part of $x(s,t)$. On the other hand, as a modification of (1·2), we have defined[2),4)]

$$\begin{aligned}\tilde{\Delta}_F[y_2(s), y_1(s)] &= \int_0^\infty d\alpha e^{-m_0^2\alpha/2}\int \mathcal{D}^8 y(s,t)\exp\{\tfrac{1}{2}D[y(s,t); D_0^\alpha]\} \\ &= \int_0^\infty d\alpha e^{-m_0^2\alpha/2} g_y[\alpha]\exp\{\tfrac{1}{2}D[y^h(s,t); D_0^\alpha]\}\,,\end{aligned} \tag{1·7}$$

where $y^h(s,t)$ is also the harmonic part of $y(s,t)$.

As D_0^α, we call for

$$R_0^\alpha = \{(s,t)\,|\, 0<s<l, 0<t<\alpha\} \tag{1·8}$$

*) Especially it should be recalled that the transposition (1·3) or (1·3') is natural from the graph-theoretical point of view.

or

$$C_0^\alpha = \{(s,t) \mid 0 \leq s \leq 2l, 0 < t < \alpha \text{ with } (0,t) \equiv (2l,t)\}, \qquad (1\cdot 8')$$

both of which reduce to the conventional Feynman path $\{t: 0<t<\alpha\}$ when $l \to 0$.

The vector $y(s,t)$ or $y^h(s,t)$ which is made use of in $(1\cdot 7)$ is also the dual-position vector which can be more naturally defined in a two-dimensional medium (see Ref. 2) and also Ref. 6)).

Incidentally the factor $\exp(-\frac{1}{2}m_0^2\alpha)$ in $(1\cdot 6)$ or $(1\cdot 7)$ is a Wick-rotated counterpart of $\exp(-\frac{1}{2}im_0^2\alpha)$ in $(1\cdot 1)$ or $(1\cdot 2)$. Let us introduce here the *inner radius* r_0 associated with $(1\cdot 8)$ or $(1\cdot 8')$ by

$$r_0 = \exp(-\pi\alpha/l). \qquad (1\cdot 9)$$

Then $\exp(-\frac{1}{2}m_0^2\alpha) = r_0^{lm_0^2/2\pi}$, so that the intercept α_0 is given by

$$\alpha_0 = -lm_0^2/2\pi .$$

(That is, if $\alpha_0 = 1$, then $m_0^2 = -2\pi/l$.)

The main object of the present paper is to investigate the possibility whether the simple symmetries which lie between the expressions $(1\cdot 1)$ and $(1\cdot 4\text{a})$ or $(1\cdot 4\text{b})$ can also be maintained in the case of the dual-resonance propagators. We deal with this problem in terms of a four-leg diagram (when $D_0^\alpha = R_0^\alpha$) or a two-leg one (when $D_0^\alpha = C_0^\alpha$). We will not, however, carry out the Fourier transformation of $(1\cdot 6)$, but we directly compare $(1\cdot 7)$ with $(1\cdot 6)$ in the same way as we compared $(1\cdot 4\text{a})$ or $(1\cdot 4\text{b})$ with $(1\cdot 1)$. As a result it will be shown that the symmetry which can hold is deeply connected with the *magic* dimensions of space-time, that is, the maximal number of dimensions for which no ghosts appear.

The present work has been inspired by a short note by Brink and Nielsen.[7] In the case in which we use the domain R_0^α, the width l plays the role of the reciprocal variable β defined by $(1\cdot 5)$ (or l/α corresponds to β, while α/l to α). This has a relation with the Jacobi imaginary transformation and so is concerned with the idea of Brink and Nielsen. On the other hand, in the case of using the domain C_0^α, such a replacement of the t-axis and the s-axis is so stringent that we can only expect the symmetry to hold under more restricted conditions.

In the next two sections, we present two auxiliaries; in § 2 we shall solve a boundary-value problem for $y^h(s,t)$ to discuss the symmetry of the Dirichlet integral factors in $(1\cdot 6)$ and $(1\cdot 7)$, and then in § 3 we consider the symmetry of the weight $g_x[\alpha]$ or $g_y[\alpha]$. Finally § 4 contains the main proposition that is concerned with the reciprocal symmetries of the full expressions of propagators.

§ 2. Dirichlet's problem for the dual-position

To obtain the explicit form of the exponential of $D[y^h(s,t); R_0^\alpha]$ in $(1\cdot 7)$, we shall first solve a boundary-value problem for the dual-position $y^h(s,t)$ when

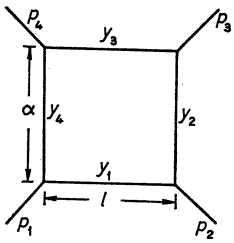

Fig. 1.

the case is as depicted in Fig. 1. Let $y^h(s,t)$ be harmonic in R_0^α and satisfy the boundary condition

$$\left.\begin{aligned} y(s,0)&=y_1 && \text{for } 0<s<l,\\ y(l,t)&=y_2 && \text{for } 0<t<\alpha,\\ y(s,\alpha)&=y_3 && \text{for } 0<s<l,\\ y(0,t)&=y_4 && \text{for } 0<t<\alpha, \end{aligned}\right\} \tag{2·1}$$

in which y_i, $i=1,2,3,4$, are constant vectors (with effective dimension δ). According to the prescription in Ref. 2), the external momenta p_1, p_2, p_3 and p_4 incident at the four corners are respectively equal to y_1-y_4, y_2-y_1, y_3-y_2 and y_4-y_3 up to a common factor. We can obtain elementarily a solution of this problem in the following form:

$$\begin{aligned} y^h(s,t) &= (1/\pi)\,\mathrm{Re}\, iy_1[\log \vartheta_1(u/2l)^2/\vartheta_1((u-l)/2l)\,\vartheta_1((u+l)/2l)]\\ &+ (1/\pi)\,\mathrm{Re}\, iy_2[\log \vartheta_1((u-l)/2l)^2/\vartheta_1((u-i\alpha-l)/2l)\,\vartheta_1((u+i\alpha-l)/2l)]\\ &- (1/\pi)\,\mathrm{Re}\, iy_3[\log \vartheta_1((u-i\alpha)/2l)^2/\vartheta_1((u-l-i\alpha)/2l)\,\vartheta_1((u+l-i\alpha)/2l)]\\ &- (1/\pi)\,\mathrm{Re}\, iy_4[\log \vartheta_1(u/2l)^2/\vartheta_1((u-i\alpha)/2l)\,\vartheta_1((u+i\alpha)/2l)], \end{aligned} \tag{2·2}^{*)}$$

where

$$u=s+it \tag{2·3}$$

and $\vartheta_1(v)$ is an abbreviation of $\vartheta_1(v|\tau)$, τ being given by

$$\tau=(\log r_0)/\pi i=i\alpha/l\,. \tag{2·4}$$

Next, let us, by means of the above solution $y^h(s,t)$, construct the *complex* position $z(s,t)$ in such a way that

$$z(s,t)=x^h(s,t)+iy^h(s,t) \tag{2·5}$$

is analytic in R_0^α. If we employ the notation

$$p_1=y_1-y_4\,,\qquad p_2=y_2-y_1\,,\qquad p_3=y_3-y_2\,,\qquad p_4=y_4-y_3\,, \tag{2·6}$$

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*) Solution of the case of (2·1) was once considered in Ref. 4), when $y_2=y_4=0$ and y_1, y_3 depend on s (see (3·10)→(3·13) in Ref. 4)).

then $z(s,t)$ given by (2·5) takes the form

$$z(s,t) = -(2/\pi)\sum_{j=1}^{4} p_j \log \vartheta_j(u/2l) \tag{2·7}$$

up to a constant term. On the other hand, it follows from Green's theorems, the Cauchy-Riemann relations and others that

$$D[y^h; R_0^\alpha] = \oint d\gamma\,(y^h, \partial y^h/\partial n) = \oint d\gamma\,(y^h, -\partial x^h/\partial\gamma) = \oint d\gamma\,(\partial y^h/\partial\gamma, x^h)$$
$$\to \oint d\gamma\,(p(\gamma), z(\gamma)), \tag{2·8}$$

where γ is an arc length of the perimeter. Hence we eventually arrive at

$$\exp\{\tfrac{1}{2}D[y^h; R_0^\alpha]\} = \prod_{k\neq j} |\vartheta_j(v_k)\vartheta_k(v_j)|^{-(p_j,p_k)/\pi}, \tag{2·9}$$

in which $k, j = 1, 2, 3, 4$ and

$$v_1 = 0\,, \qquad v_2 = 1/2\,, \qquad v_3 = 1/2 + \tau/2\,, \qquad v_4 = \tau/2\,. \tag{2·10}$$

Let us now turn to the case where D_0^α is C_0^α. However, the necessary solution of the boundary-value problem has already been listed in Ref. 8). Let us tentatively pick out a simple case, as in Ref. 5), where

p_0 is incident to $(s=0, t=0)$

and

$-p_\alpha$ is incident to $(s=2m\ (\leqq 2l), t=\alpha)$.

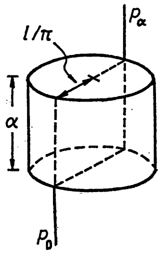

Fig. 2.

Then, corresponding to (2·9) one finds

$$\exp\left\{\frac{1}{2}D[y^h; C_0^\alpha]\right\} = \left|\vartheta_1\left(\frac{m}{l}+\frac{\tau}{2}\middle|\tau\right)\vartheta_1\left(-\frac{m}{l}+\frac{\tau}{2}\middle|\tau\right)\right|^{-(p_0,p_\alpha)/2\pi}. \tag{2·11}$$ *)

It should be remarked that in either case we have the theta-functions of the form $\vartheta_j((a+ib)/l|i\alpha/l)$. This function is known to satisfy the following relation:**)

*) The rhs is further rewritten as

$$|-r_0^{-1/2}\ \vartheta_4(m/l|\log r_0/\pi i)\vartheta_4(-m/l|\log r_0/\pi i)|^{-(p_0,p_\alpha)/2\pi}\,.$$

**) See, for example, Ref. 9).

$$\left|\vartheta_j\left(\frac{a+ib}{l}\middle|\frac{i\alpha}{l}\right)\right|=\left|\left(\frac{i\alpha}{l}\right)^{-1/2}\cdot e^{-\pi i((a+ib)/l)^2\cdot l/2\alpha}\right|\cdot\left|\vartheta_j\left(\frac{b-ia}{\alpha}\middle|\frac{il}{\alpha}\right)\right|, \tag{2·12}$$

which fully indicates that l should be a reciprocal of α. It is hence convenient to introduce a "reciprocal radius" r_∞ by

$$r_\infty=\exp(-\pi l/\alpha), \tag{2·13}$$

so that

$$\log r_0\cdot\log r_\infty=\pi^2. \tag{2·14}$$ *)

Applying (2·12) to (2·9), we thus obtain the symmetrical relation

$$\prod_{k\neq j}|\vartheta_j(v_k|(\log r_0)/\pi i)\vartheta_k(v_j|(\log r_0)/\pi i|^{-(p_j,p_k)/\pi}$$
$$=|(\log r_0)/\pi i|^{\Sigma(p_j,p_k)/\pi}\prod_{k\neq j}|\vartheta_j(v_k^*|(\log r_\infty)/\pi i)\vartheta_k(v_j^*|(\log r_\infty)/\pi i)|^{-(p_j^*,p_k^*)/\pi}, \tag{2·15}$$

in which

$$\left.\begin{array}{l} v_1^*=0,\quad v_2^*=1/2,\quad v_3^*=1/2-(\log r_\infty)/2\pi i,\quad v_4^*=-(\log r_\infty)/2\pi i \\ p_1^*=p_1,\quad p_2^*=p_4,\quad p_3^*=p_3,\quad p_4^*=p_2 \end{array}\right\} \tag{2·16}$$

and *we have assumed* $p_j^2=0$.

Similarly, applying (2·12) to (2·11), we can prove the relation

$$\left|\vartheta_1\left(\frac{m}{l}+\frac{\log r_0}{2\pi i}\middle|\frac{\log r_0}{\pi i}\right)\vartheta_1\left(-\frac{m}{l}+\frac{\log r_0}{2\pi i}\middle|\frac{\log r_0}{\pi i}\right)\right|^{-(p_0,p_\alpha)/2\pi}$$
$$=|(\log r_0)/\pi i|^{(p_0,p_\alpha)/2\pi}(r_\infty^{2m^2/l^2}r_0^{-1/2})^{-(p_0,p_\alpha)/2\pi}$$
$$\times\left|\vartheta_1\left(\frac{m}{i\alpha}+\frac{1}{2}\middle|\frac{\log r_\infty}{\pi i}\right)\vartheta_1\left(-\frac{m}{i\alpha}+\frac{1}{2}\middle|\frac{\log r_\infty}{\pi i}\right)\right|^{-(p_0,p_\alpha)/2\pi}. \tag{2·17}$$

This is less symmetrical except when

$$2m=l. \tag{2·18}$$

Suppose (2·18) is satisfied. Then (2·17) now reads

$$\left|\vartheta_1\left(\frac{1}{2}+\frac{\log r_0}{2\pi i}\middle|\frac{\log r_0}{\pi i}\right)\vartheta_1\left(-\frac{1}{2}+\frac{\log r_0}{2\pi i}\middle|\frac{\log r_0}{\pi i}\right)\right|^{-(p_0,p_\alpha)/2\pi}$$
$$=|(\log r_0)/\pi i|^{(p_0,p_\alpha)2\pi}(r_\infty^{1/2}r_0^{-1/2})^{-(p_0,p_\alpha)/2\pi}$$
$$\times\left|\vartheta_1\left(-\frac{\log r_\infty}{2\pi i}+\frac{1}{2}\middle|\frac{\log r_\infty}{\pi i}\right)\vartheta_1\left(\frac{\log r_\infty}{2\pi i}+\frac{1}{2}\middle|\frac{\log r_\infty}{\pi i}\right)^{-(p_0,p_\alpha)/2\pi}. \tag{2·17'}$$

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*) Remark that the relation (2·14) is identical with $R_1\cdot R_2=\pi^2$ of Brink and Nielsen.[7] We have adopted in the present paper the view-point that α is a proper-time. However, as in Ref. 6), the view that $\log r_0$ or α is a "specific resistance" is also possible.

The restriction (2·18) is intuitively natural since the domain is cylindrical and (2·18) says that the two points to which p_0 and p_α are incident should be symmetrically opposite.

Before closing this section, we remark that the expression of (2·9) continues to be the same even if we set $y_4=y_2=0$ or $y_1=y_3=0$ (though the momentum conservation should be more restricted). Suppose $y_4=y_2=0$, as in Ref. 4). Then we can regard $y^h(s,t)$ as an eigenvalue of the following dual-position operator

$$Y(s,t)=(1/\sqrt{\pi})\sum_{\substack{\nu=-\infty\\ \nu\neq 0}}^{\infty}(B_\nu/\nu)\sin\nu(\pi/l)s\cdot e^{-\nu(\pi/l)t}, \tag{2·19}$$

where the hermitian coefficients B_ν are assumed to satisfy the quantum condition

$$[B_\nu{}^\mu, B_{-\nu'}^{\mu'}]_-=i\nu\delta_{\nu\nu'}g^{\mu\mu'}. \tag{2·20}$$

In this case we can write

$$g_y[\alpha]\exp\{\tfrac{1}{2}D[y^h;R_0{}^\alpha]\}=\langle y_3;\alpha|y_1;0\rangle, \tag{2·21}$$

in which $|y;t\rangle$ is a simultaneous eigenstate of $Y(s,t)$ for all $0<s<l$.

Let us set $y_3=y_1=0$ in addition to $y_2=y_4=0$ in (2·21). Then we are led to the following characterization of $g_y[\alpha]$:

$$g_y[\alpha]=\langle y_3=0;\alpha|y_1=0;0\rangle. \tag{2·22}$$

§ 3. Determination of the weights $g_x[\alpha]$ and $g_y[\alpha]$

Alternatively we can construct, as we did in Ref. 3), a quantum mechanics starting with the position operator $X^\mu(s,t)$ defined by

$$X(s,t)=(1/\sqrt{l})(x_0+p_0t)-(1/\sqrt{\pi})\sum_{\substack{\nu=-\infty\\ \nu\neq 0}}^{\infty}(B_\nu/\nu)\cdot\cos\nu(\pi/l)s\cdot e^{-\nu(\pi/l)t}. \tag{3·1}$$

Then, instead of (2·22), we have the relation

$$g_x[\alpha]=\langle x_3=0;\alpha|x_1=0;0\rangle, \tag{3·2}$$

where $g_x[\alpha]$ is the one already given at (1·6). Here the numbering system follows the one in § 2. It should be noted in advance that the Schrödinger equation for $\langle x;t|$ implies

$$\langle x;\alpha|=\langle x;0|\exp(-i\alpha H_B), \tag{3·3}$$

where

$$H_B=-p_0^2/2-(\pi/2l)\sum_{\substack{\nu=-\infty\\ \nu\neq 0}}^{\infty}(B_\nu,B_{-\nu}). \tag{3·3'}$$

We need further the (indefinite) occupation number states $|n_\nu{}^\mu\rangle$ of $B_\nu{}^\mu B_{-\nu}^\mu/\nu$ constructed from the "vacuum" $|0\rangle$ defined by

$$B_\nu{}^\mu|0\rangle=0\,. \qquad (\nu>0)$$

Let us first remark that there exists a factor

$$\langle x_{03}=0\,;\alpha|x_{01}=0\,;0\rangle$$

included in (3·2) where $|x_0;t\rangle$ is an eigenstate of (x_0+p_0t). Since

$$\langle x_{03}=0\,;\alpha|\,(x_0+p_0t)\,|\rangle=0\,, \tag{3·4}$$

we can verify

$$\langle x_{03}=0\,;\alpha|x_{01}=0\,;0\rangle=\text{const}\lim_{x_{01}\to 0}\alpha^{-\delta_0/2}\exp(-x_{01}^2/\alpha)=\text{const}\;\alpha^{-\delta_0/2}, \tag{3·5}$$

where δ_0 indicates the effective dimension of x_0.

Next we should calculate the factor

$$\begin{aligned}&\langle x_\nu;0|\exp\{-(i\pi\alpha/2l)\,[B_\nu B_{-\nu}+B_{-\nu}B_\nu]\}\,|x_\nu';0\rangle\\&=\sum_{n_\nu}\langle x_\nu;0|n_\nu\rangle\langle n_\nu|x_\nu';0\rangle\,\exp[-(\pi\alpha/l)\,\nu n_\nu],\end{aligned} \tag{3·6}$$

where $|x_\nu;0\rangle$ is an eigenstate of $(B_\nu-B_{-\nu})/\nu$. Notice then that

$$\langle x_\nu;0|\,[B_\nu B_{-\nu}+B_{-\nu}B_\nu]\,|n_\nu\rangle=2i\nu n_\nu\langle x_\nu;0|n_\nu\rangle \tag{3·7}$$

proves to be an Hermite differential equation whose solution is $\langle x_\nu;0|n_\nu\rangle$. Hence the expression

$$\begin{aligned}\langle x_\nu;0|n\rangle&=\text{const}\cdot\lim_{x\to 0}H_n(x)/(2^n n!)^{1/2}\\&=\text{const}\begin{cases}0 & \text{for } n=\text{odd}\,,\\(-1/2)^{n/2}n!^{1/2}/(n/2)! & \text{for } n=\text{even}\,,\end{cases}\end{aligned} \tag{3·8}$$

because of the behaviour of the Hermite polynomial $H_n(x)$.[10] It follows then that the rhs of (3·6) equals

$$\sum_{k=1}^{\infty}(2k)!/(2^k k!)^2 e^{-2(\pi/l)\alpha\nu k}=\{1-\exp[-2\pi(\alpha/l)\,\nu]\}^{-1/2}. \tag{3·9}$$

This is of course identical with a result by Brink and Nielsen.[7] Thus we arrive at

$$g_x[\alpha]=\text{const}\,(\alpha/l)^{-\delta_0/2}\prod_{\nu=1}^{\infty}\{1-\exp[-2\pi(\alpha/l)\,\nu]\}^{-\delta/2}, \tag{3·10}$$

where δ is the effective dimension of B_ν (which we provisionally distinguish from δ_0).

Now that we have (3·10), then, using the change of variable (1·9), we can eventually rewrite (1·6) as

$$\begin{aligned}\varDelta_F[x_3(s),x_1(s)]=\text{const}&\int_0^1 dr_0 r_0{}^{-\alpha_0-1}(-\log r_0)^{-\delta_0/2}\prod_{\nu=1}^{\infty}(1-r_0{}^{2\nu})^{-\delta/2}\\&\times\exp\{-\tfrac{1}{2}D[x^h(s,t)\,;R_0{}^\alpha]\},\end{aligned} \tag{3·11}$$

which, we stress again, corresponds to (1·1).

Now it is apparent that if we follow a similar line of manipulation, we can also put forward the following expression for $\tilde{\Delta}_F[y_3, y_1]$:

$$\tilde{\Delta}_F[y_3, y_1] = \text{const} \int_0^1 dr_0 r_0^{-\alpha_0-1} \prod_{\nu=1}^{\infty} (1-r_0^{2\nu})^{-\delta/2} \exp\{\tfrac{1}{2}D[y^h(s,t); R_0^\alpha]\}, \quad (3\cdot 12\text{a})$$

which actually corresponds to (1·4a). The tentative reason why we have erased the factor $(-\log r_0)^{-\delta_0/2}$ is that $Y(s,t)$ given by (2·19) is free from the zero-mode term, and this corresponds to the fact that in the integrand of (1·4a) missing is the factor α^{-2} which, however, the integrand of (1·1) possesses.

Our choice of $Y(s,t)$ of (2·19), however, depends on the boundary data $y_4 = y_2 = 0$. Therefore if we more generally suppose the finite boundary values on both sides of R_0^α, the situation becomes somewhat different. In fact this alternation seems necessary if one wants to bring about the expression of $\tilde{\Delta}_F[y_3, y_1]$ corresponding to (1·4b). In this case we should make use of the variable l/α instead of α/l, as well as r_∞ defined by (2·13). Then it may be appropriate to put forward the following expression

$$\tilde{\Delta}_F[y_3, y_1] = \text{const} \int_0^1 dr_\infty r_\infty^{-\alpha_0-1} (-\log r_\infty)^{-\delta_0/2} \prod_{\nu=1}^{\infty} (1-r_\infty^{2\nu})^{-\delta/2}$$
$$\times \exp\{\tfrac{1}{2}D[y^h(s,t); R_0^\alpha]\}\,|_{r_0 \to r_\infty} \quad (3\cdot 12\text{b})$$

which now corresponds to (1·4b). This time we have taken into account the zero-mode term and $Y(s,t)$ should turn to be of the form

$$Y(s,t) = (1/\sqrt{l})(y_0 + q_0 s) + (1/\sqrt{\pi}) \sum_{\substack{\nu=-\infty \\ \nu \neq 0}}^{\infty} (B_\nu'/\nu) \cdot \sin \nu(\pi/\alpha) t \cdot e^{-\nu(\pi/\alpha)s}. \quad (3\cdot 13)$$

Here y_0 and B_ν' are non-commutable with q_0 and $B'_{-\nu}$ respectively in the same way the set x_0, p_0, B_ν and $B_{-\nu}$ satisfy.

Let us next turn to the pomeron-propagator case (when $D_0^\alpha = C_0^\alpha$). The expression of $\Delta_F[x_\alpha(s), x_0(s)]$, however, remains nearly the same as (3·11) except for the fact that $\prod(1-r_0^{2\nu})^{-\delta/2}$ in (3·11) should be replaced by $\prod(1-r_0^{2\nu})^{-\delta}$. This is because the number of coefficient operators of the following $X(s,t)$ for the pomeron is doubled:[5)]

$$X(s,t) = (1/\sqrt{2l})(x_0 + p_0 t)$$
$$- (1/\sqrt{2\pi}) \sum_{\substack{\nu=-\infty \\ \nu \neq 0}}^{\infty} [(A_\nu/\nu) \sin \nu(\pi/l)s + (B_\nu/\nu) \cos \nu(\pi/l)s] e^{-\nu(\pi/l)t}.$$
$$(3\cdot 14)$$

In accordance, $\tilde{\Delta}_F[p_\alpha, p_0]$*) corresponding to (3·12a) should take the form

*) We write $\tilde{\Delta}_F[p_\alpha, p_0]$ for $\tilde{\Delta}_F[y(s)]$ of (1·7), since simple relations between momenta and dual positions are missing in non-planar cases.

$$\tilde{\Delta}_F[p_\alpha, p_0] = \text{const} \int_0^1 dr_0 r_0^{-\alpha_0 - 1} \prod_{\nu=1}^{\infty} (1 - r_0^{2\nu})^{-\delta} r_0^{-\varepsilon(p_0, p_\alpha)/4\pi}$$

$$\times \left| \vartheta_1\left(\frac{m}{l} + \frac{\log r_0}{2\pi i} \middle| \frac{\log r_0}{\pi i} \right) \vartheta_1\left(-\frac{m}{l} + \frac{\log r_0}{2\pi i} \middle| \frac{\log r_0}{\pi i} \right) \right|^{-(p_0, p_\alpha)/2\pi}, \tag{3·15a}$$

while the one corresponding to (3·12b) should be of the form

$$\tilde{\Delta}_F[p_\alpha, p_0] = \text{const} \int_0^1 dr_\infty r_\infty^{-\alpha_0 - 1} (-\log r_\infty)^{-\delta_0/2} \prod_{\nu=1}^{\infty} (1 - r_\infty)^{-\delta} r_\infty^{-\varepsilon(p_0, p_\alpha)/4\pi}$$

$$\times \left| \vartheta_1\left(\frac{m}{l} + \frac{\log r_\infty}{2\pi i} \middle| \frac{\log r_\infty}{\pi i} \right) \vartheta_1\left(-\frac{m}{l} + \frac{\log r_\infty}{2\pi i} \middle| \frac{\log r_\infty}{\pi i} \right) \right|^{-(p_0, p_\alpha)/2\pi}. \tag{3·15b}$$

Note that we have inserted by hand the factor $r_0^{-\varepsilon(p_0, p_\alpha)/4\pi}$ in (3·15a) and $r_\infty^{-\varepsilon(p_0, p_\alpha)/4\pi}$ in (3·15b), whereas a straight-forward result from the definition implies $\varepsilon = 0$. However, as will be demonstrated below, if we are allowed to put $\varepsilon = 1$, then we shall be led to a more symmetrical result. Let us remark that, when $\varepsilon = 1$, this extra factor cancels the $r_0 \to 0$ or $r_\infty \to 0$ singularity arising from the ϑ_1-functions in the integrand.

§ 4. Reciprocal symmetries of the propagator

We are now in a position to consider possible symmetries appearing on full expressions of the dual-resonance propagators. Let us first try to exchange the integration variable in (3·12a) from r_0 to r_∞ by use of (2·14):

$$\tilde{\Delta}_F[y_3, y_1] = \text{const} \int_0^1 dr_\infty r_\infty^{-1} r_0^{-\alpha_0} (\log r_\infty)^{-2} \prod_{\nu=1}^{\infty} (1 - r_0^{2\nu})^{-\delta/2}$$

$$\times \prod_{k \neq j} |\vartheta_j(v_k) \vartheta_k(v_j)|^{-(p_j, p_k)/\pi}, \tag{4·1}$$

in which $r_0 = \exp(\pi^2/\log r_\infty)$. Let us next exploit the relation (2·15) and the well-known formula

$$\prod_{\nu=1}^{\infty} (1 - r_0^{2\nu})^{-1} = \pi^{1/2} r_0^{1/12} r_\infty^{-1/12} (\log r_\infty)^{-1/2} \prod_{\nu=1}^{\infty} (1 - r_\infty^{2\nu})^{-1} \tag{4·2}$$

on the rhs of (4·1). Then

$$\tilde{\Delta}_F[y_3, y_1] = \text{const} \int_0^1 dr_\infty r_\infty^{-(\delta/24) - 1} r_0^{(\delta/24) - \alpha_0} (\log r_\infty)^{-2 - (\delta/4)} \prod_{\nu=1}^{\infty} (1 - r_\infty^{2\nu})^{-\delta/2}$$

$$\times |\log r_0/\pi i|^{\Sigma(p_j, p_k)/\pi} \prod_{j \neq k} \left| \vartheta_j\left(v_k^* \middle| \frac{\log r_\infty}{\pi i} \right) \vartheta_k\left(v_j^* \middle| \frac{\log r_\infty}{\pi i} \right) \right|^{-(p_j^*, p_k^*)/\pi}. \tag{4·3}$$

By our requirement that the integrand of (4·3) should conform to the integrand of (3·11) (in the same way as the integrand of (1·4b) does to that of (1·1)), we obtain the equations

$$\left.\begin{aligned} &-\alpha_0+(\delta/24)=0\,,\\ &2+(\delta/4)+\textstyle\sum(p_j,\,p_k)/\pi=\delta_0/2\,.\end{aligned}\right\} \tag{4·4}$$

Incidentally, if the first equation of (4·4) is satisfied, we have the factor $r_\infty^{-\alpha_0-1}$ in (4·3) and this corresponds to the fact that the factor $\exp(-im_0^2\alpha/2)$ in (1·4a) becomes $\exp(-im_0^2/2\beta)$ in (1·4b). Now suppose all $p_j=0$. Then we have $\delta=24\alpha_0$ and $\delta_0=4+12\alpha_0$. If $\alpha_0=1$, we shall obtain $\delta=24$ and $\delta_0=16$, the last being compatible with our recent result $c(=\delta_0/2)=8$ in Ref. 11). If we instead put $\delta=\delta_0$, then

$$\sum_{j\neq k}(p_j,\,p_k)=4\pi\,. \tag{4·5}^{*)}$$

Let us next choose the second expression (3·12b). In this case, the requirement we should impose turns to be that if we write (3·12b) in terms of the r_0-variable, then the factor in the integrand made up of $-\log r_0$ should disappear in order to produce only a pole singularity in momentum space. This is also the case when (1·4b) is transcribed back to (1·2) with the use of (1·5). On exploiting (2·15) and (4·2), we readily have

$$\begin{aligned}\tilde{\Delta}_F[y_1,\,y_3]=\text{const}\int_0^1 dr_0 r_0^{-(\delta/24)-1}r_\infty^{-\alpha_0+(\delta/24)}(\log r_0)^{-2+(\delta_0/2)-(\delta/4)}\prod_{\nu=1}^{\infty}(1-r_0^{2\nu})^{-\delta/2}\\ \times\left|\frac{\log r_\infty}{\pi i}\right|^{\Sigma(p_j,\,p_k)/\pi}\prod_{k\neq j}\left|\vartheta_j\left(v_k\left|\frac{\log r_0}{\pi i}\right.\right)\vartheta_k\left(v_j\left|\frac{\log r_0}{\pi i}\right.\right)\right|^{-(p_j,\,p_k)/\pi}.\end{aligned} \tag{4·6}$$

Hence we obtain

$$\left.\begin{aligned} &-\alpha_0+(\delta/24)=0\,,\\ &-2+(\delta_0/2)-(\delta/4)-\textstyle\sum(p_j,\,p_k)/\pi=0\,,\end{aligned}\right\} \tag{4·7}$$

which, however, is exactly the same with (4·4). This duality was also alluded in Ref. 11).

We can formally carry out a similar procedure in the case of pomeron propagator. Let us first rewrite $\tilde{\Delta}_F[p_\alpha,\,p_0]$ of (3·15a) for $2m=l$, (2·18), as

$$\begin{aligned}\tilde{\Delta}_F[p_\alpha,\,p_0]\\ &=\text{const}\int_0^1 dr_\infty r_\infty^{-1}r_0^{-\alpha_0}(\log r_\infty)^2\prod_{\nu=1}^{\infty}(1-r_0^{2\nu})^{-\delta}r_0^{-\varepsilon(p_0,\,p_\alpha)/4\pi}\\ &\quad\times\left|\vartheta_1\left(\frac{1}{2}+\frac{\log r_0}{2\pi i}\left|\frac{\log r_0}{\pi i}\right.\right)\vartheta_1\left(-\frac{1}{2}+\frac{\log r_0}{2\pi i}\left|\frac{\log r_0}{\pi i}\right.\right)\right|^{-(p_0,\,p_\alpha)/2\pi}\\ &=\text{const}\int_0^1 dr_\infty r_\infty^{-1-(\delta/12)-(p_0,\,p_\alpha)/4\pi}r_0^{-\alpha_0+(\delta/12)+(1-\varepsilon)(p_0,\,p_\alpha)/4\pi}\\ &\quad\times(\log r_\infty)^{-2-(\delta/2)}\prod_{\nu=1}^{\infty}(1-r_\infty^{2\nu})^{-\delta}\left|\frac{\log r_0}{\pi i}\right|^{(p_0,\,p_\alpha)2\pi}\end{aligned}$$

*) Fairlie and Roberts[12] once remarked that a direct application of the Weierstrauss condition (of the Plateau problem) implies $\sum(p_j,\,p_k)=0$.

$$\times\left|\vartheta_1\left(\frac{1}{2}-\frac{\log r_\infty}{2\pi i}\middle|\frac{\log r_\infty}{\pi i}\right)\vartheta_1\left(\frac{1}{2}+\frac{\log r_\infty}{2\pi i}\middle|\frac{\log r_\infty}{\pi i}\right)\right|^{-(p_0,p_\alpha)/2\pi}, \qquad (4\cdot 8)$$

in which we have also used $(4\cdot 2)$ and $(2\cdot 17')$. By a direct inspection, we shall obtain, in place of $(4\cdot 4)$, the following condition:

$$\left.\begin{aligned}-\alpha_0+(\delta/12)+(1-\varepsilon)(p_0,p_\alpha)/4\pi=0\,,\\ 2+(\delta/2)+(p_0,p_\alpha)/2\pi=\delta_0/2\,.\end{aligned}\right\} \qquad (4\cdot 9)$$

It should, however, be noticed that we have necessarily the factor $r_\infty^{-(p_0,p_\alpha)/4\pi}$ in $(4\cdot 8)$ which cancels another reciprocal factor arising from the ϑ_1-functions. Hence, by comparison with $(3\cdot 15\mathrm{a})$, we should be led to a more symmetrical situation when we choose $\varepsilon=1$. Then the first equation of $(4\cdot 9)$ implies that $\alpha_0=\delta/12$. Therefore it follows from $\alpha_0=2$ that $\delta=24$, which is again the magic number of dimensions. The second equation tells us that if $\delta=\delta_0$, then $(p_0,p_\alpha)=-4\pi$.

The final concern is with the rhs of $(3\cdot 15\mathrm{b})$ rewritten for $2m=l$ as

$$\tilde{\Delta}_F[p_\alpha,p_0]=\text{const}\int_0^1 dr_0 r_0^{-1-(\delta/12)-(p_0,p_\alpha)/4\pi} r_\infty^{-\alpha_0+(\delta/12)+(1-\varepsilon)(p_0,p_\alpha)/4\pi}$$

$$\times(\log r_0)^{-2+(\delta_0/2)-(\delta/2)}\prod_{\nu=1}^{\infty}(1-r_0^{2\nu})^{-\delta}$$

$$\times\left|\frac{\log r_\infty}{\pi i}\right|^{(p_0,p_\alpha)/2\pi}\left|\vartheta_1\left(-\frac{\log r_0}{2\pi i}+\frac{1}{2}\middle|\frac{\log r_0}{\pi i}\right)\vartheta_1\left(\frac{\log r_0}{2\pi i}+\frac{1}{2}\middle|\frac{\log r_0}{\pi i}\right)\right|^{-(p_0,p_\alpha)/2\pi}. \qquad (4\cdot 10)$$

It is however apparent that the condition derived from the requirement that the multiple of $-\log r_0$ in the integrand should be zero is exactly identical to $(4\cdot 9)$.

We have thus far given an affirmative answer to the possibility of a reciprocal symmetry which can formally exist between the representations of dual-resonance propagators. As to its physical implications or its impact on further development of theoretical sides of the models, the present author has not yet any opinion. However, the relation $\log r_0\cdot\log r_\infty=\pi^2$, $(2\cdot 14)$, on which the reciprocal symmetry naively depends, seems to suggest that this symmetry is of new character somewhat different from the Veneziano duality which may be considered as depending on the relation of the type $r_0+r_0'=1$. In this connection, it should be recalled that a parallelism rather strongly lies between the non-linear relation $(2\cdot 14)$ and the relation which defines the temperature of the *dual net* of the two-dimensional Ising system, that is, the relation of the form $\sinh(J/kT)\cdot\sinh(J/kT^*)=1$.*) On the other side, this tells us a possibility of a thermodynamical description of our symmetry (for example, it may be conjectured that the critical case of the Douglas condition for the pomeron propagator[5] can be interpreted as corresponding to a Curie point).

*) This is also alluded by Cremmer and Scherk.[13]

Finally it is also interesting to note that our symmetry is truly reminiscent of the so-called Nelson symmetry[14] which, roughly speaking, says that an interchange of the time and the length of one-volume leaves an inner product invariant. Nowadays it is known that Nelson's symmetry has a trend of bringing a vast economy in the area of the so-called $P(\phi)_2$-field theory.*) Our reciprocal symmetry is also expected to work as a tool to bring on a new interpretation and a further characterization of the dual-resonance model.

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*) Refer, for example, to Ref. 15).