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Functional Approach to the Critical Dimension in Veneziano's Model

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We give an explicit derivation of the critical dimension of space-time associated with the Veneziano model starting from the path-integral expression based on a Dirichlet-type boundary-value problem. It is shown that our expression for four-leg amplitudes happens to reduce to the beta-function formula of Veneziano only when the effective number of transverse dimensions is 24. Up to the goal, the transformation properties are not self-evident since our representation has a batch of the theta functions in the integrand. Our next concern is therefore with a discussion how the Möbius invariance or the reciprocal invariance is connected with the critical dimensionality.

§ 1. Introduction

In a previous paper[1] (referred to as [I] in what follows) we showed that there exists an apparent resemblance between the integral representations of the dual-resonance propagators in momentum space and those in position space when the number of the effective dimensions of space-time is twenty-four. Briefly speaking, this symmetry holds when we replace the proper-time β by the reciprocal one $1/\beta$, so that we termed this property the reciprocal symmetry.

In the present paper, we wish to ask the question as to whether we can impose a stronger requirement that the momentum space representations of the Veneziano amplitudes are *invariant* in themselves against the reciprocity transformations. We only deal with the four-point case, but the answer to be concluded appears to be general and we see that they are really reciprocally invariant if the effective number of transverse dimensions is again 24.

The clue to our plan is firstly provided by a reconstruction of the amplitude in the language of the Feynman path-integrations, and then it will be proved that the amplitude reduces to the Veneziano formula only for the critical number of dimensions, and thirdly it will be concluded that the obtained formula is invariant under the reciprocity transformations. To treat the dimension explicitly we adopt a rectangular domain as the domain of the field variables (dual-position d-vectors) and make extensive use of Jacobi's $\vartheta_j(z|\tau)$-functions. This was also the case in [I].

It should be recalled in advance that, while $\vartheta_j(z|\tau)$ is an elliptic function with respect to z, it also behaves as an elliptic modular function with respect to τ.[2),3)] Our reciprocity is just concerned with the change $\tau \to -1/\tau$. In this sense, our proposition can be thought of as discussing the critical dimension by requiring the integrand to be invariant against the modular group generated by $\tau \to -1/\tau$. On the other hand, transformations with respect to τ can explicitly be related to those with respect to z through the so-called Jacobi's imaginary transformations. Hence it is also possible to restate the above proposition from the viewpoint of the Möbius transformation concernig z. It is for this reason that we are in fact able to verify the proposition only by the requirement that the formula is Möbius invariant, before we appeal to the reciprocity.

Incidentally it should be remarked that the critical value of dimension will be specified in connection with the invariant volume elements, so that our proposition can be a parallel to a result by Mandelstam,[4)] though the proceduce to the end appears quite different.

In the second section we prepare, following [I], the functional integral expression for the four-point amplitudes, which is reasoned as corresponding to a generalization of the conventional Feynman propagator. This will be established within a framework of the Dirichlet problem with the aid of the dual-position vectors $y(s,t)$.[5)] In § 3 it is proved that the expression constructed in § 2 agrees truly with the Veneziano amplitude iff the number δ of transverse degrees of freedom is twenty-four (and the intercept α_0 is unity). The Veneziano formula is known to be Möbius invariant. However, our expressions up to the goal are written down by the theta functions so that the transformation properties are somewhat obscure. Therefore, in the first half of § 4, we try to explicate its relation with the familiar homographic transformations of the unit-disk by defining the Koba-Nielsen variables. In the latter half of § 4 we introduce explicitly the reciprocity transformations and shall confirm that our conclusions in the text can also be drawn from the sole requirement of the reciprocity invariance.

§ 2. Preliminaries

In this section we prepare the expression for the amplitude with which we should start. The line of thought adopted follows closely that of the previous work, so that almost all materials we here present are those recapitulated from [I]. There is, however, one delicate difference in that we explicitly pay attention to the condition that incoming particles are tachyons.

The fundamental tool which figures in the first stage is the dual-position function $y(s,t)$ which is a d-dimensional Minkowski vector.*) Let us assume that the domain of $y(s,t)$ is

$$R_0{}^{\beta} = \{s, t \,|\, 0 < s < l,\ 0 < t < \beta\} \tag{2·1}$$

*) The Lorentz product of y_1 and y_2 is denoted by $(y_1, y_2) = y_1^{(0)} y_2^{(0)} - \sum_{j=1}^{d-1} y_1^{(j)} y_2^{(j)}$. We write however y^2 for (y, y).

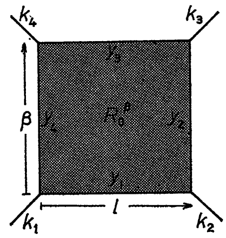

Fig. 1.

and that $y(s,t)$ takes the following values at the boundary ∂R_0^β:

$$\left.\begin{array}{lll} y(s,0)=y_1\,, & y(s,\beta)=y_3 & \text{for}\quad 0<s<l\,, \\ y(l,t)=y_2\,, & y(0,t)=y_4 & \text{for}\quad 0<t<\beta\,, \end{array}\right\} \tag{2·2}$$

in which y_j, $j=1, 2, 3, 4$, are implied to be constant. This last assumption infers that the case we treat is that of four particles coming into the corners; their momenta k_j being given by the relations

$$\left.\begin{array}{ll} 2\alpha'\pi k_1=y_1-y_4\,, & 2\alpha'\pi k_3=y_3-y_2\,, \\ 2\alpha'\pi k_2=y_2-y_1\,, & 2\alpha'\pi k_4=y_4-y_3\,. \end{array}\right\} \tag{2·3}*)$$

The Feynman kernel associated with this process is of the form

$$\begin{aligned} \widetilde{K}(y_1, y_2, y_3, y_4; \beta) &= \int_{R_0^\beta} \mathcal{D}y(s,t) \exp\left\{\frac{1}{4\pi\alpha'}D[y(s,t); R_0^\beta]\right\} \\ &= g(\beta) \exp\left\{\frac{1}{4\pi\alpha'}D[y^h(s,t); R_0^\beta]\right\}, \end{aligned} \tag{2·4}$$

in which $D[y \text{ or } y^h; R_0^\beta]$ is the Dirichlet integral with respect to y or y^h. Here $y^h(s,t)$ is the dual-position *harmonic* in R_0^β and is assumed to enjoy the same boundary value as $y(s,t)$. Consequently if we write

$$y(s,t)=y^h(s,t)+y^0(s,t)\,, \tag{2·5}$$

then $y^0(s,t)$ is a deviation which must vanish at ∂R_0^β. In terms of y^0, $g(\beta)$ in (2·4) is written as

$$g(\beta)=\int_{R_0^\beta} \mathcal{D}y^0(s,t) \exp\left\{\frac{1}{4\pi\alpha'}D[y^0(s,t); R_0^\beta]\right\}. \tag{2·6}$$

From this, it follows easily that

$$g(\beta)=\text{const}\prod_{\nu=1}^{\infty}(1-r_0^{2\nu})^{-8/2}\,, \tag{2·7}$$

where r_0 is given by

*) α' can be identified with the Regge slope. For the present, it is a resistance if k_j is regarded as the current and y_j as a voltage.

$$r_0 = e^{-\pi\beta/l} \tag{2·8}$$

and δ is the effective dimension; that is, $\delta = d - e$, if e is the number of the dimensions which do not contribute due to an implicit presence of gauges: We however do not address ourselves to e itself in this paper.

As verified in [I], the harmonic $y^h(s,t)$ which is subjected to the boundary condition (2·2) is solved to be the imaginary part of

$$z(s,t) = -4\alpha' \sum_{j=1}^{4} k_j \log \vartheta_j(u/2l), \tag{2·9}$$

where $u = s + it$, and $\vartheta_j(u/2l)$ is an abbreviation of Jacobi's theta function $\vartheta_j(u/2l|\log r_0/\pi i)$.

In [I] we assumed that k_j are all light-like in order to prevent the Dirichlet integral from diverging. This time, however, we must take the condition

$$\alpha' k_j^2 = -\alpha_0 \tag{2·10}$$

into account. Hence we are forced to remove the divergence by hand. It is easy to see that the following is the divergent terms in $D[y^h; R_0^\beta]$:

$$\begin{aligned} &-2\alpha' k_j^2 \log|\vartheta_j(v_j+\varepsilon)| \\ &= \begin{cases} -2\alpha' k_j^2 \log|\vartheta_1'(0)| - 2\alpha' k_j^2 \log \varepsilon & \text{for } j=1,2, \\ -2\alpha' k_j^2 \log|r_0^{-1/4}\vartheta_1'(0)| - 2\alpha' k_j^2 \log \varepsilon & \text{for } j=3,4, \end{cases} \end{aligned} \tag{2·11}$$

when $\varepsilon \to 0$. Here the v_j's denote

$$v_1 = 0,\ v_2 = \frac{1}{2},\ v_3 = \frac{1}{2} + \frac{\log r_0}{2\pi i},\ v_4 = \frac{\log r_0}{2\pi i}. \tag{2·12}$$

On reference to the usual procedure,*) we readily recognize that it is the $\log \varepsilon$ terms of (2·11) that we should erase. Doing this operation, we find that the formula which takes the place of (2·19) in [I] turns out to be

$$\begin{aligned} \exp\left\{\frac{1}{4\pi\alpha'} D[y^h; R_0^\beta]\right\}\Bigg|_{(2\cdot 3)} &= r_0^{-\alpha_0} \vartheta_1'(0)^{8\alpha_0} \prod_{i>j} |\vartheta_i(v_j)\vartheta_j(v_i)|^{-2\alpha'(k_i,k_j)} \\ &= \prod_{i>j} \left| \frac{\vartheta_i(v_j)\vartheta_j(v_i)}{c_i\vartheta_1'(0) c_j \vartheta_1'(0)} \right|^{-2\alpha'(k_i,k_j)}, \end{aligned} \tag{2·13}$$ **)

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*) In the conventional operator formalism, one drops the diverging terms by invoking the normal ordering:

$$:\exp\left[\sqrt{2\alpha'} \sum_{\nu=1}^{\infty} \frac{(k, a_\nu^\dagger - a_\nu)}{\sqrt{\nu}}\right]: = \exp\left[\sqrt{2\alpha'} \sum_{\nu=1}^{\infty} \frac{(k, a_\nu^\dagger)}{\sqrt{\nu}}\right] \exp\left[-\sqrt{2\alpha'} \sum_{\nu=1}^{\infty} \frac{(k, a_\nu)}{\sqrt{\nu}}\right],$$

which otherwise gives rise to the exponent of

$$2\alpha' \left[\sum_{\nu=1}^{\infty} \frac{(k, a_\nu^\dagger)}{\sqrt{\nu}}, \sum_{\nu=1}^{\infty} \frac{(k, a_\nu)}{\sqrt{\nu}}\right]_- = -2\alpha' k^2 \log \varepsilon.$$

**) Note that (2·9) can also be written as

$$z(s,t) = -4\alpha' \sum_{j=1}^{4} k_j \log[\vartheta_j(u/2l)/\vartheta_1'(0)]$$

owing to the coservation $\sum k_j = 0$. When $k_j^2 = 0$, the common factor $\vartheta_1'(0)$ does not contribute to (2·13), so that (2·19) in [I] remains valid.

Recall, on the other hand, that the factor $\vartheta_1'(0)$ in (2·13) reveals itself already in the one-loop diagram case. See, for example, Eq. (3·3) in Ref. 6).

where

$$c_1=-c_2=1\,,\quad c_3=c_4=ir_0^{-1/4}\,. \tag{2·14}$$

To see the last equivalence of (2·13), we have only to consult

$$\prod_{i>j}|c_ic_j|^{2\alpha'(k_i,k_j)}=r_0^{-\alpha_0}\,. \tag{2·15}$$

Summing up, we have thus found that the amplitude

$$\int_0^\infty d\beta e^{+\alpha_0\pi\beta/l}\widetilde{K}(y_1,y_2,y_3,y_4;\beta)|_{(2\cdot3)} \tag{2·16}$$

which is originally a generalization of the Feynman propagator,[5] can be paraphrased as follows:

$$V(k)=\text{const}\int_0^1 dr_0 r_0^{-1-\alpha_0}\prod_{\nu=1}^{\infty}(1-r_0^{2\nu})^{-\delta/2}r_0^{-\alpha_0}\vartheta_1'(0)^{8\alpha_0}$$
$$\times\prod_{i>j}|\vartheta_i(v_j)\vartheta_j(v_i)|^{-2\alpha'(k_i,k_j)}\,. \tag{2·17}$$

Then our programme of relating (2·17) with the Veneziano formula starts.

§ 3. Critical dimension and the beta-function amplitudes

Now that (2·17) is obtained, the argument to the conclusion is straightforward and simple. The key to the method lies in rewriting further the integrand in (2·17) in terms of $\vartheta_2(0)$, $\vartheta_3(0)$ and $\vartheta_4(0)$: By virtue of $\vartheta_1(v_2)=\vartheta_2(0)$, $\vartheta_1(v_3)=r_0^{-1/4}\cdot\vartheta_3(0)$ and so on, we can firstly verify that

$$\prod_{i>j}|\vartheta_i(v_j)\vartheta_j(v_i)|^{-2\alpha'(k_i,k_j)}=r_0^{\alpha_0}|\vartheta_3(0)^4|^{-2}|\vartheta_2(0)^4\vartheta_4(0)^4|^{1-\alpha_0}$$
$$\times\left|\frac{\vartheta_2(0)^4}{\vartheta_3(0)^4}\right|^{-1-\alpha_0-\alpha's}\left|\frac{\vartheta_4(0)^4}{\vartheta_3(0)^4}\right|^{-1-\alpha_0-\alpha't}, \tag{3·1}$$

where

$$s=(k_1+k_2)^2\quad\text{and}\quad t=(k_2+k_3)^2\,. \tag{3·2}$$

On the other hand, widely-known formulae tell us that

$$\prod_{\nu=1}^{\infty}(1-r_0^{2\nu})^{-\delta/2}=2^{\delta/6}r_0^{\delta/24}(\vartheta_2(0)^4\vartheta_3(0)^4\vartheta_4(0)^4)^{-\delta/24} \tag{3·3}$$

and

$$\vartheta_1'(0)^{8\alpha_0}=\pi^{8\alpha_0}(\vartheta_2(0)^4\vartheta_3(0)^4\vartheta_4(0)^4)^{2\alpha_0}\,. \tag{3·4}$$

These bring Eq. (2·17) to the form

$$V(k)=\text{const}\int_0^1 dr_0\, r_0^{-1-\alpha_0+\delta/24}\vartheta_3(0)^{-\delta/6-8+8\alpha_0}(\vartheta_2(0)\vartheta_4(0))^{-\delta/6+4+4\alpha_0}$$
$$\times\left(\frac{\vartheta_2(0)^4}{\vartheta_3(0)^4}\right)^{-1-\alpha_0-\alpha's}\left(\frac{\vartheta_4(0)^4}{\vartheta_3(0)^4}\right)^{-1-\alpha_0-\alpha't}. \tag{3·5}$$

Let us write

$$\lambda=\frac{\vartheta_2(0)^4}{\vartheta_3(0)^4}. \tag{3·6}$$

Then it follows from the relation $\vartheta_2(0)^4+\vartheta_4(0)^4=\vartheta_3(0)^4$ that

$$\frac{\vartheta_4(0)^4}{\vartheta_3(0)^4}=1-\lambda. \tag{3·7}$$

It should be recalled that λ depends explicitly on r_0 since $\vartheta_j(0)\equiv\vartheta_j(0|\log r_0/\pi i)$. Therefore we should next be concerned with the change of variable from r_0 to λ. Incidentally, λ is never a new notation, and its use can be traced back to Weierstrass et al. (about 1850).[3] In fact, such properties of λ as analyticity, automorphism and so on have ever been exhausted.[3),2)] What follows is a somewhat elementary result: If we make the change $\tau\equiv\log r_0/\pi i$ to $\tau+1$, then λ becomes $\lambda(\lambda-1)^{-1}$ and also if we do the change

$$\tau\to-\frac{1}{\tau} \tag{3·8}$$

then

$$\lambda\to1-\lambda. \tag{3·9}$$

Hence

$$\left.\begin{aligned}&\lambda=0 \quad \text{when} \quad r_0=0\\ &\text{and}\\ &\lambda=1 \quad \text{when} \quad r_0=1,\end{aligned}\right\} \tag{3·10}$$

though it is known that λ is analytic and nowhere takes 0 or 1 in the upper half plane of $\tau=\log r_0/\pi i$.

The formula which we next need is provided by

$$\frac{dr_0}{r_0}=d\lambda\frac{\vartheta_3(0)^4}{\vartheta_2(0)^4\vartheta_4(0)^4}, \tag{3·11}$$

which however we missed to find in literature, so that we shall record a proof of (3·11) in the Appendix.

In the light of (3·10) and (3·11), we can eventually rewrite (3·5) as follows:

$$V(k)=\text{const}\int_0^1 d\lambda\, r_0^{-\alpha_0+8/24}\vartheta_3(0)^{-8/6-4+8\alpha_0}$$
$$\times(\vartheta_2(0)\vartheta_4(0))^{-8/6+4\alpha_0}\lambda^{-1-\alpha_0-\alpha' s}(1-\lambda)^{-1-\alpha_0-\alpha' t}, \tag{3·12}$$

which is apparently equivalent to the Beta-function amplitude of Veneziano:

$$V(k)=\text{const}\,B(-\alpha' s-\alpha_0,\ -\alpha' t-\alpha_0) \tag{3·13}$$

only if

$$-\alpha_0+\delta/24=0\,,\quad -\delta/6-4+8\alpha_0=0\,, \tag{3·14}$$

whence

$$\alpha_0=1\,,\quad \delta=24\,. \tag{3·15}$$

We have thus shown that the four-point amplitude (2·17) produced by the functional integration agrees with the Veneziano amplitude only when the number of the effective dimensions is just 24.*) For other values of δ, the integration measure happens to have some factors which may give rise to singularities other than poles and destroy the Möbius invariance.

It is, however, somewhat implicit how the Möbius transformation works if we make exclusive use of the ϑ_j-functions to deal with the integrand and the integration measure. We therefore try, in the first half of the next section, to reveal the relation between the Möbius invariance of the Koba-Nielsen kernel and the possible automorphism of the kernel described by the ϑ_j-functions. To this end, we differently rewrite Eq. (2·17), when subjected to (3·15), in the form:

$$V(k)=\text{const}\int_0^1(dr_0/r_0)\,\vartheta_3(0)^4 W(k;\log r_0/\pi i) \tag{3·16}$$

with

$$W\left(k;\frac{\log r_0}{\pi i}\right)=\left(\frac{\vartheta_2(0)^4\vartheta_4(0)^4}{r_0}\right)^{\alpha_0}\prod_{i>j}|\vartheta_i(v_j)\vartheta_j(v_i)|^{-2\alpha'(k_i,k_j)}. \tag{3·17}$$

§ 4. The Möbius invariance or the reciprocal invariance of the Veneziano formula

In the first half of this section we discuss the invariancy of the integrand by means of Möbius transformations, while the second half is concerned with the same invariancy from the viewpoint of the reciprocity foreshadowed in [I].

4·1 *The Möbius invariance*

Let ρ_j be the Koba-Nielsen variables on a unit circle ($|\rho_j|=1$). Then it is conventional to pick out the factor

$$\prod_{i>j}|\rho_i-\rho_j|^{-2\alpha'(k_i,k_j)}\prod_{j=1}^{4}|\rho_{j+1}-\rho_j|^{\alpha_0} \tag{4·1}$$

as being the integrand invariant against the group of automorphism of the unit circle:

$$\rho_j\to\frac{\rho_j-\rho_0}{1-\rho_j\bar{\rho}_0}e^{i\varphi}\,, \tag{4·2}$$

*) The condition $\alpha_0=1$ is no more than a consistent output since it is rather well known from the outset that our functional integral has a connection with the Veneziano model only when $\alpha_0=1$.

where $|\rho_0|<1$ and φ is real. Let us here recall the following conformal transformations which send a unit disk into the rectangular domain $R_0{}^{\beta}$:[7]

$$\rho=\frac{\vartheta_1((u-u_0)/2l)\vartheta_1((u+u_0)/2l)}{\vartheta_1((u-\bar{u}_0)/2l)\vartheta_1((u+\bar{u}_0)/2l)}e^{i\phi}, \tag{4·3}$$

in which $u=s+it$ as before and $u_0(\in R_0{}^{\beta})$ and ϕ denote also three parameters. From this it follows that

$$\rho_j=e^{i\phi}\vartheta_j(u_0/2l)^2/\vartheta_j(\bar{u}_0/2l)^2, \tag{4·4}$$

for ρ_j should correspond to v_j which characterizes j-corner of $R_0{}^{\beta}$.

Now we try to transform the invariant (4·1) by (4·4): Firstly we find

$$\prod_{i>j}|\rho_i-\rho_j|^{-2\alpha'(k_i,k_j)}$$

$$=\frac{\prod_{i>j}|\vartheta_i(v_j)\vartheta_j(v_i)|^{-2\alpha'(k_i,k_j)}|\vartheta_1((u+\bar{u}_0)/2l)\vartheta_1((u-\bar{u}_0)/2l)|^{-2\alpha'\sum_{i>j}(k_i,k_j)}}{\prod_{j=1}^{4}|\vartheta_j(\bar{u}_0/2l)|^{2\alpha' k_j{}^2}}, \tag{4·5}$$

in which we have restored v_j again. The derivation of (4·5) is somewhat complicated but straightforward by use of the so-called addition formulae of the theta-functions.[8]

Secondly we find

$$\prod_{j=1}^{4}|\rho_{j+1}-\rho_j|^{\alpha_0}$$

$$=\frac{\prod_{j=1}^{4}|\vartheta_{j+1}(v_j)\vartheta_j(v_{j+1})|^{\alpha_0}|\vartheta_1((u_0+\bar{u}_0)/2l)\vartheta_1((u_0-\bar{u}_0)/2l)|^{4\alpha_0}}{\prod_{j=1}^{4}|\vartheta_j(\bar{u}_0/2l)|^{2\alpha_0}}. \tag{4·6}$$

Equations (4·5) and (4·6) combine to reveal that (4·1) equals

$$\prod_{i>j}|\vartheta_i(v_j)\vartheta_j(v_i)|^{-2\alpha'(k_i,k_j)}\prod_{j=1}^{4}|\vartheta_{j+1}(v_j)\vartheta_j(v_{j+1})|^{\alpha_0}, \tag{4·7}$$

which however is identical with $W(k;\log r_0/\pi i)$ defined by (3·17) since

$$\prod_{j=1}^{4}|\vartheta_{j+1}(v_j)\vartheta_j(v_{j+1})|^{\alpha_0}=r_0{}^{-\alpha_0}|\vartheta_2(0)^4\vartheta_4(0)^4|^{\alpha_0}. \tag{4·8}$$

We are thus convinced of the fact that $W(k;\log r_0/\pi i)$ plays the same role as (4·1) and is Möbius invariant (against the homographic transformations of the form (4·2)). This assertion may of course be directly proved by means of the λ function, but the correspondence of $W(k;\log r_0/\pi i)$ to (4·1) seems more informative for further outlook.

The integration measure remaining in (3·16) is equal to $\lambda^{-1}(1-\lambda)^{-1}d\lambda$, which is obviously invariant under $\lambda\to 1-\lambda$. In the light of (3·8) and (3·9), this transformation corresponds to $\tau\to-1/\tau$, which however is just the reciprocity transformation once featured in [I]. We therefore discuss next the same problem from this point of view.

4·2 *Reciprocal invariance*

By reciprocity, we originally meant the associated properties of dual-resonance quantities which show up when one interchanges l and β with each other (or l/β by β/l). This is somewhat concerned with a symmetry which appears when one tries to interchange the s-axis and the t-axis, and may be of practical importance when one is forced to compare the quantities for $\beta\to 0$ with those for $\beta\to\infty$. In view of this, let us define

$$r_\infty = e^{-\pi l/\beta} \tag{4·9}$$

as a companion of (2·8). Then

$$\log r_0 \cdot \log r_\infty = \pi^2 . \tag{4·10}$$ *)

Jacobi's theta functions $\vartheta_j(z|\tau)$ with $\tau=\log r_0/\pi i$ are related to $\vartheta_{j*}(z^*|\tau^*)$ where $\tau^*=-1/\tau=\log r_\infty/\pi i$ through the so-called Jacobi's imaginary transformations:

$$\left|\vartheta_j\left(\frac{a+ib}{l}\middle|\frac{\log r_0}{\pi i}\right)\right| = \left(\frac{\pi}{-\log r_0}\right)^{1/2} r_0^{\pi^2(a^2-b^2)/l^2}\left|\vartheta_{j*}\left(\frac{b-ia}{\beta}\middle|\frac{\log r_\infty}{\pi i}\right)\right|, \tag{4·11}$$

where j^* stands for 1, 4, 3, 2 when $j=1$, 2, 3, 4 respectively. By use of (4·11) we can verify

$$\begin{aligned}\prod_{i>j}&\left|\vartheta_i\left(v_j\middle|\frac{\log r_0}{\pi i}\right)\vartheta_j\left(v_i\middle|\frac{\log r_0}{\pi i}\right)\right|^{-2\alpha'(k_i,k_j)}\\ &=|(r_0/r_\infty)^{1/4}(\log r_0/\pi)|^{-\alpha'\sum_{j=1}^{4}k_j^2}\\ &\quad\times\prod_{i>j}\left|\vartheta_{i*}\left(v_j^*\middle|\frac{\log r_\infty}{\pi i}\right)\vartheta_{j*}\left(v_i^*\middle|\frac{\log r_\infty}{\pi i}\right)\right|^{-2\alpha'(k_i,k_j)},\end{aligned} \tag{4·12}$$ **)

where

$$v_1^*=0,\ v_2^*=\frac{1}{2},\ \ v_3^*=\frac{1}{2}-\frac{\log r_\infty}{2\pi i},\ \ v_4^*=-\frac{\log r_\infty}{2\pi i}. \tag{4·13}$$

Similarly we have

$$\begin{aligned}&\left|\frac{\vartheta_2(0|\log r_0/\pi i)^4\vartheta_4(0|\log r_0/\pi i)^4}{r_0}\right|^{\alpha_0}\\ &=|(r_0/r_\infty)^{1/4}(\log r_0/\pi)|^{-4\alpha_0}\left|\frac{\vartheta_2(0|\log r_\infty/\pi i)^4\vartheta_4(0|\log r_\infty/\pi i)^4}{r_\infty}\right|^{\alpha_0}.\end{aligned} \tag{4·14}$$

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*) On this relation, it was once physically interpreted by Brink and Nielsen[9] that $\alpha'\log r_0$ and $\alpha'\log r_\infty$ are "resistances" of the rectangular medium.

**) Equation (4·12) is a detailed version of (2·15) in [I] for the case $k_j^2\neq 0$. Numbering in (2·15) of [I] is asymmetrical since we have miscited the foregoing (2·12) (which is corrected here as (4·11)).

Hence it follows that

$$W\left(k;\frac{\log r_0}{\pi i}\right)=W\left(k;\frac{\log r_\infty}{\pi i}\right). \tag{4·15}$$

On the other hand, it can also be shown

$$\frac{dr_0}{r_0}\vartheta_3\left(0\left|\frac{\log r_0}{\pi i}\right.\right)^4=-\frac{dr_\infty}{r_\infty}\vartheta_3\left(0\left|\frac{\log r_\infty}{\pi i}\right.\right)^4 \tag{4·16}$$

so that the integrand including the integration measure is invariant against the reciprocity transformations. (To prove (4·16), it suffices only to combine (4·10) and (4·11) for $j=j^*=3$.) Any way it has now been clearly turned out that we cannot establish the Möbius invariance or the reciprocity invariance for other values of δ than 24 if we retain the form of (3·5) or (3·12).

§ 5. Concluding remarks

We have thus far succeeded in showing how the critical number of space-time dimensions which underlies the Veneziano amplitudes is connected with the Möbius invariance or the reciprocity invariance. Conversely, our success justifies our preference for the functional integral approach based on the dual-positions, and also justifies us in associating a rectangular domain with the beta-function amplitudes.*) In fact, we consider that the number δ is explicit in our framework because of the rectangular domain through the volume element factor $\prod_{\nu=1}^{\infty}(1-r_0^{2\nu})^{-\delta/2}$. On the other hand, we are not led to make mention of e itself where $e=d-\delta$; there will be no gain until the projection to the physical states is explicitly taken into account.

Our consideration has been restricted to a four-leg process. However it may be possible to extend it to the case where many particles come into the rectangular medium, if we modify the boundary condition (2·2). In this case, the conformal transformation (4·3) is still available, while (4·4) should correspondingly be complicated. To this end, the manipulations in § 4·1 may be of practical help.

Appendix

We here verify (3·11). Since $\lambda(=\vartheta_2(0)^4/\vartheta_3(0)^4)$ is explicitly given by

$$\lambda=16r_0\frac{\prod_{\nu=1}^{\infty}(1+r_0^{2\nu})^8}{\prod_{\nu=1}^{\infty}(1+r_0^{2\nu-1})^8}, \tag{A·1}$$

we may derictly obtain $d\lambda/dr_0$. We however prove (3·11) here less labourously

*) The present author once thought of a picture[10] in which λ is identified with $\mathrm{sn}^2(Z,k)$, instead of $\vartheta_2(0)^4/\vartheta_3(0)^4$. In that case $1-\lambda=\mathrm{cn}^2(Z,k)$. It must also be possible to carry out a similar argument if one is able to determine the volume element.

with the aid of some known formulae. Let us first recall the relation[8)]

$$r_0 = \exp(-\pi K'/K)\,, \tag{A·2}$$

where K and K' are the complete elliptic integrals of the first kind given by

$$\left.\begin{aligned} K &= \frac{\pi}{2} F\left(\frac{1}{2}, \frac{1}{2}, 1; \lambda\right) = \frac{\pi}{2} \vartheta_3(0|\log r_0/\pi i)^2\,, \\ K' &= \frac{\pi}{2} F\left(\frac{1}{2}, \frac{1}{2}, 1; 1-\lambda\right) = \frac{\pi}{2} \vartheta_3(0|\log r_\infty/\pi i)^2\,, \end{aligned}\right\} \tag{A·3}$$

λ being originally termed as k^2. On the other hand we have

$$\begin{aligned} &\frac{d}{dz} F\left(\frac{1}{2}, \frac{1}{2}, 1; z\right) \\ &\quad = -F\left(\frac{1}{2}, \frac{1}{2}, 1; z\right)\Big/ 2z + F\left(-\frac{1}{2}, \frac{1}{2}, 1; z\right)\Big/ 2z(1-z)\,. \end{aligned} \tag{A·4}$$

Thus we can verify

$$\frac{d}{d\lambda}\left(\frac{K'}{K}\right) = \frac{1}{2\lambda(1-\lambda)}\left[\frac{K'}{K} - \frac{E}{K} - \frac{K'E}{K^2}\right], \tag{A·5}$$

where E and E' stand for the complete elliptic integrals of the second kind given by

$$E = \frac{\pi}{2} F\left(-\frac{1}{2}, \frac{1}{2}, 1; \lambda\right), \quad E' = \frac{\pi}{2} F\left(-\frac{1}{2}, \frac{1}{2}, 1; 1-\lambda\right). \tag{A·6}$$

It is fortunately known that $EK' + E'K - KK' = \pi/2$ (Legendre's relation).[8)] Hence

$$d\left(\frac{K'}{K}\right) = -\frac{\pi}{4} \cdot \frac{1}{K^2} \cdot \frac{d\lambda}{\lambda(1-\lambda)}\,. \tag{A·7}$$

Recalling (A·2) and (A·3) as well as (3·6) and (3·7), we eventually obtain (3·11).

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