Progress of Theoretical Physics, Vol. 58, No. 5, November 1977

A Unified Einstein-Yukawa Theory

George L. MURPHY*)

Department of Physics, The University of Western Australia, Nedlands Western Australia 6009

(Received February 22, 1977)

Following Schrödinger's hint that the charge independence of nuclear forces is analogous to the mass-independence of the gravitational interaction required by the equivalence principle, we construct a non-Riemannian geometry in which the worldlines of particles moving in combined gravitational and scalar meson fields are the paths of space-time. A geometric Lagrangian which yields the coupled Einstein-Yukawa equations is given. A characteristic "order of magnitude" difficulty of geometric unified field theories can be examined here in closed form.

§ 1. Introduction

In the first years after the initial success of Einstein's geometric theory of gravitation, many theorists believed in the possibility of a unified geometric theory of gravitation and electrodynamics, at that time the only known interactions.[1] A major factor in discouraging most physicists from working on geometric unified field theories was the discovery of new types of interaction in the nucleus, and Yukawa's[2] recognition that new fields had to be introduced to describe nuclear forces. It is, indeed, difficult to imagine any historical scenario, consistent with the existence of life, in which the gravitational and electromagnetic interactions would not have been studied fairly thoroughly before the discovery of the strong and weak interactions. However, it is at least logically possible to ignore electromagnetic (and weak) interactions, and to try to formulate a very simple model in which at least part of the strong interaction is described in terms of geometry, as is gravitation in general relativity.

There must be some physical motivation to give direction to our mathematics. The main thing that motivated Einstein's geometrization of gravitation was the fact that the acceleration of a test particle in a gravitational field does not depend on that particle's mass or composition. Schrödinger[3] pointed out that the charge-independence of the strong interaction between nucleons suggests that the nuclear force is closer to the gravitational interaction in this regard than is the electromagnetic force. This suggestion was made before the proliferation of known hadrons, and the analogy should not be pushed too far. (For example, the interaction between a nucleon and a Λ^0 obviously cannot be the same as that between two nucleons: the vertex $\Lambda \to \Lambda + \pi$ cannot occur, because the Λ has isospin zero, and the π has isospin one.) Nevertheless, it seems worthwhile to take Schröd-

*) Present address: Dept. of Physics, Luther College, Decorah, Iowa 52101.

inger's suggestion as a first step toward unification of gravity and nuclear forces.

The model which will be presented in the next section is rather artificial, and one may well wonder what value it has. It is certainly of interest to show that part of the strong interaction can be geometrized, but it is perhaps of more importance that we can construct a unified geometric theory which is not beset by difficulties of calculation or interpretation (as are the non-symmetric theories of Einstein and Schrödinger[4]), so that the physical content is clearly revealed.

In particular, we encounter a difficulty with the *strength* of the "strong" interaction in this model, even though the *form* of the field equations is satisfactory. This difficulty will be discussed in the third section. It is probably characteristic of geometric unified field theories, and reasons will be given for believing it to be non-fatal.

§ 2. Geometric representation of the pion field

The equation of paths in a non-Riemannian space with the symmetric connection

$$\Gamma^{\alpha}_{\beta\gamma} = \left\{ {\alpha \atop \beta\gamma} \right\} + C^{\alpha}{}_{\beta\gamma} \tag{1}$$

is

$$U^{\mu} U^{\sigma}{}_{|\mu} = -C^{\sigma}{}_{\alpha\beta} U^{\alpha} U^{\beta} . \tag{2}$$

Here the Christoffel affinity $\left\{ {\alpha \atop \beta\gamma} \right\}$ is formed in the usual way from the metric $g_{\mu\nu}$, and the vertical stroke indicates covariant differentiation with respect to this affinity. $C^{\alpha}{}_{\beta\gamma}$ is a tensor, symmetric in β and γ. U^{μ} is the unit tangent to the path ($U^{\mu}U_{\mu} = +1$), to be identified with the four-velocity of a nucleon. The term on the right of (2) is a four-acceleration produced by the non-gravitational interaction.

In particular, we may take[5]

$$C^{\alpha}{}_{\beta\gamma} = -g_{\beta\gamma}\phi^{,\alpha} + \tfrac{1}{2}(\delta^{\alpha}_{\beta}\phi_{,\gamma} + \delta^{\alpha}_{\gamma}\phi_{,\beta}) \tag{3}$$

with ϕ a scalar or pseudoscalar field. ($C^{\alpha}{}_{\beta\gamma}$ is, of course, a pseudotensor in the latter case, which corresponds to actual pions. Presumably we must then call $\Gamma^{\alpha}_{\beta\gamma}$ a pseudo-affinity. I shall continue to use the words scalar, etc., since I shall not be considering reflections here.) Note that (3) does *not* give Weyl's affinity for the vector field $\phi_{,\alpha}$, and that this approach to the scalar field is therefore different from the recent one of Rothwell.[6]

With the choice (3), the path equation (2) becomes

$$U^{\mu} U^{\sigma}{}_{|\mu} = (g^{\mu\sigma} - U^{\mu} U^{\sigma})\phi_{,\mu} , \tag{4}$$

which is the equation of motion for a particle in a scalar field. (A mass and coupling constant will be supplied in the next section.) In particular, we find that $U^{\mu} U_{\sigma} U^{\sigma}{}_{|\mu} = 0$, so that the four-velocity and four-acceleration calculated with the Christoffel affinity are orthogonal.

We now need field equations for the coupled $g_{\mu\nu}$ and ϕ fields. The Ricci tensor for our affinity is

$$R_{\mu\nu}=\Gamma^{\sigma}_{\mu\sigma,\nu}-\Gamma^{\sigma}_{\mu\nu,\sigma}+\Gamma^{\sigma}_{\mu\rho}\Gamma^{\rho}_{\sigma\nu}-\Gamma^{\sigma}_{\mu\nu}\Gamma^{\rho}_{\sigma\rho}$$
$$=B_{\mu\nu}+\tfrac{1}{2}\phi_{|\mu\nu}+g_{\mu\nu}\phi^{|\alpha}{}_{\alpha}+\tfrac{1}{4}\phi_{|\mu}\phi_{|\nu}+\tfrac{1}{2}g_{\mu\nu}\phi^{|\alpha}\phi_{|\alpha}\,, \tag{5}$$

where $B_{\mu\nu}$ is the Ricci tensor formed from the Christoffel affinity. We note in passing that $R_{\mu\nu}$ is symmetric.

The curvature scalar, which will form the main part of our Lagrangian, is

$$R=B+\frac{9}{2}\phi^{|\alpha}{}_{\alpha}+\frac{9}{4}\phi^{|\alpha}\phi_{|\alpha}\,. \tag{6}$$

Since ϕ is a geometric object—it, along with $g_{\mu\nu}$, determines the affinity—our requirement that the theory be purely geometric will allow us to combine any scalar function of ϕ and its derivatives with R in a Lagrangian.

It is first convenient to subtract the term $(9/2)\phi^{|\alpha}{}_{\alpha}$ from (6). Actually it is a matter of comparative indifference whether this is done or not, for when multiplied by $\sqrt{-g}$, this term becomes a pure divergence, $((9/2)\sqrt{-g}\phi^{,\alpha})_{,\alpha}$. It will thus contribute nothing to the field equations if the variations $\delta\phi$ in Hamilton's principle have vanishing derivatives at the boundary, as well as vanishing there themselves. But it is simplest to just eliminate this term.

We may also add a cosmological term to R, but this need not be a constant, since functions of ϕ may be used. If the field equations for ϕ are to be linear and homogeneous, this term must have the form $-(2\Lambda+9m^2\phi^2/4)$. Here Λ is the usual cosmological constant, and m is another constant with the units of an inverse length. (ϕ is dimensionless.) The Lagrangian is then

$$\boldsymbol{L}=\sqrt{-g}\left[R-\frac{9}{2}\phi^{|\alpha}{}_{\alpha}-2\Lambda-\frac{9}{4}m^2\phi^2\right]$$
$$=\sqrt{-g}\left[B-2\Lambda+\frac{9}{4}(\phi_{,\alpha}\phi_{,\beta}g^{\alpha\beta}-m^2\phi^2)\right]. \tag{7}$$

Variation of $g_{\mu\nu}$ will give the Einstein equations

$$B_{\mu\nu}-\frac{1}{2}Bg_{\mu\nu}=-\Lambda g_{\mu\nu}-\frac{9}{4}\left[\phi_{,\mu}\phi_{,\nu}-\frac{1}{2}(\phi_{,\alpha}\phi^{,\alpha}-m^2\phi^2)g_{\mu\nu}\right], \tag{8}$$

and variation of ϕ yields the Klein-Gordon equations

$$\phi^{|\alpha}{}_{\alpha}+m^2\phi=0\,. \tag{9}$$

One could easily construct more complicated theories. For example, the term $-(9/4)m^2\phi^2$ in the Lagrangian could be replaced by a more general function of ϕ, giving rise to a non-linear self-coupling for the scalar field. One could also multiply R by functions of ϕ, rather than simply add such functions. This would introduce features similar to those of the Brans-Dicke theory.[7]

It should be noted that, when the Lagrangian (7) is written explicitly in terms of $g_{\mu\nu}$ and $C^{\sigma}{}_{\tau\rho}$, the last term is non-local since $C^{\sigma}{}_{\tau\rho}$ depends only on the gradient of ϕ. This non-locality would not arise for a massless scalar field ($m=0$).

With the connection given by (1) and (3), the change in length of a vector A^{μ} on parallel displacement through dx^{σ} is[7]

$$\delta(A^{\mu}A_{\mu})=C^{\sigma}_{\mu\nu}A^{\mu}A^{\nu}dx_{\sigma}$$

and this vanishes when $dx^{\sigma}=A^{\sigma}d\lambda$, with λ some scalar parameter. In particular, the length of the unit tangent U^{σ} to a curve does not change along the curve. However, the length of a vector is not, in general, integrable. We cannot yet say how this should effect the structure of nucleons in our theory, for their structure is determined by the same field ϕ which is responsible for the changes in length.

A correct treatment of the nuclear interaction must, of course, be in accord with quantum theory. The ϕ field can be quantized in accord with the usual rules, but in principle this requires also that the gravitational field be quantized, and this is much more difficult.

It has been shown that nucleons, treated as structureless classical particles whose worldlines are the paths of space-time, will interact via a Yukawa potential as well as by means of Einsteinian gravitation. One naturally wonders if it is possible to introduce a more realistic description of nucleons as Dirac particles. In order to do this in a satisfactory way, it would be necessary to use a more general geometry in which a spinor field would be a natural element, just as the paths of space-time are a natural element of the present geometry.

It would be possible to introduce a spinor field as a non-geometric entity ψ obeying a Dirac-like equation. The "natural" form of such an equation contains the Fock-Ivanenko coefficients, with terms proportional to the connection:[8]

$$i\gamma^{\mu}(\psi_{,\mu}+\Gamma_{\mu}\psi)+M\psi=0 \tag{10}$$

with

$$\Gamma_{\mu}=\frac{1}{8}[\gamma^{\nu}\gamma_{\mu,\nu}-\gamma_{\mu,\nu}\gamma^{\nu}-\Gamma^{\rho}_{\mu\nu}(\gamma^{\nu}\gamma_{\rho}-\gamma_{\rho}\gamma^{\nu})].$$

Thus in addition to the terms which occur with a Riemannian background there will be a term proportional to $\gamma^{\mu}\phi_{,\mu}\psi$, arising from $C^{\sigma}_{\mu\nu}$, in the Dirac equation, representing a derivative coupling. But it is possible, though artificial, to add extra terms to (10) "by hand" to cancel out this term involving $\phi_{,\mu}$ and replace it with one proportional to $\gamma^5\phi\psi$, which would be in better accord with the properties of the actual nuclear interactions.

§ 3. Numerical values

The parameter m is independent of the other constants in the model, and we may choose it to be the inverse of the pion's reduced Compton wavelength. Λ may be chosen to fit astrophysical data.[9]

ϕ is proportional to the actual value of the pseudoscalar field Φ which one usually considers in particle physics, and we write $\phi = k\Phi$. In order for (4) to give the correct classical equations of motion for nucleons of mass M, coupled to the Φ field with hadronic charge g, we must have $k = g/Mc^2$. To obtain the correct contribution of the Φ field's energy-momentum tensor to the Einstein equations, we must have, from (8), $(9/4)k^2 = 8\pi G/c^4$. If we put $\beta = g^2/\hbar c$, we see that these two conditions on k imply

$$M/\sqrt{\beta} \approx \sqrt{\frac{\hbar c}{G}}, \tag{11}$$

to within a factor of order unity. With $\beta \approx 1$, M must be on the order of the Planck mass, $2.2 \times 10^{-5} g$. This is far from being true for actual nucleons. To say this in another way, since (11) is equivalent to $g^2 = GM^2$, the scalar field interaction of the nucleons has the same strength as gravity. (The fact that black holes with the Planck mass are "strongly interacting particles" has been noted by Treder.[10])

Thus while our equations have the correct *form*, the numerical values seem quite wrong. This is not surprising, for both the gravitational and nuclear interaction arises here from different parts of the same object $\Gamma^{\mu}_{\nu\sigma}$, and will thus have the same strength. Similar difficulties can occur in non-symmetric theories, in which gravity and electromagnetism are represented by the symmetric and skew-symmetric parts of a second rank tensor.

This result should not be too discouraging, for it refers only to the bare mass and coupling constant. Quantization and renormalization must be effected before our equations for the strong interaction can be of any value. It is quite possible that these procedures, together with the effects of quantum gravity, could make the interaction of the right order of magnitude.

It is worth noting that particle masses on the order of the Planck mass also occur in recent gauge attempts to unify the strong, electromagnetic and weak interactions.[11]

References

1) See, e.g., A. S. Eddington, *The Mathematical Theory of Relativity*, 2nd ed. (Cambridge U. Press, Cambridge, 1924), Chap. VII and notes 13 and 14, for an account of some of the early work on unified field theories.
2) H. Yukawa, Proc. Phys.-Math. Soc. Japan **17** (1935), 48.
3) E. Schrödinger, Proc. Roy. Irish Acad. **49A** (1944), 275, **51A** (1947), 147.
4) A. Einstein, *The Meaning of Relativity*, 5th ed. (Princeton U. Press, Princeton, N. J., 1955), Appendix II.
E. Schrödinger, *Space-Time Structure*, (Cambridge U. Press, Cambridge, 1963), Chap. 12.
5) G. L. Murphy, Phys. Rev. **D11** (1975), 2752, Section VI. The sign of ϕ here differs from that in the present paper.
6) P. L. Rothwell, Nuovo Cim. **30B** (1975), 267.
7) C. Brans and R. H. Dicke, Phys. Rev. **124** (1961), 925.
8) J. L. Anderson, *Principles of Relativity Physics* (Academic Press, New York, 1967), pp. 358-360.
9) See, e.g., J. E. Gunn and B. M. Tinsley, Nature **257** (1975), 454.
10) H. J. Treder, Ann. der Phys. **32** (1975), 238.
11) H. Georgi, H. R. Quinn and S. Weinberg, Phys. Rev. Letters **33** (1974), 451.