

**Functional Evaluation of the Dual Partition Functions***—How Do We “Hear the Shape of a Drum” in Dual Models?—*

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It is shown how the partition functions of dual strings are evaluated in the framework of the functional formalism. The functional integrals are made explicit by the use of the Kronecker limit formulae, let alone well defined thanks to the zeta-function regularizations.

**§ 1. Introduction**

The partition function of a given dual model is an arithmetical tool to specify how many states with a definite mass squared of the resonances the dual model has, and for instance it readily informs us how the degeneracy increases as the energy does.<sup>\*)</sup> It is true that any dual model is not determined solely by the partition of the mass spectra, but any dual partition function well features the mathematical characteristics of its respective model: We may be thus allowed to say that the variety of the partition functions well reflects the variety of the dual models. It is therefore always preferable to have a well-established foundation for the formulation of dual partition functions.

Our aim in this paper is to give a consistent way of describing and calculating the partition functions of the known dual models in the framework of the so-called functional formalism. We share thus our objective in common with those in Refs. 2) and 3), but it will turn out that our line of calculation still has some value of its own.

In § 2, we review a lemma due to Kac<sup>4)</sup> on a characteristic of annulus domains, which will form a starting background for our way of description. In § 3, we begin to deal with the partition function of Veneziano's orbital model. In the first subsection it is defined as a functional integral and is converted to be compactly written by the values of the zeta-function along the line of Hawking.<sup>5)</sup> In the second subsection the values are explicitly evaluated by the use of several known formulae including Kronecker's first limit formula. The partition function of the orbital string proves without overcharge to be a modular form of weight  $-\delta/2$ , where  $\delta$  denotes the effective dimension of space-time. The divergence problem of the “zero-point energies” is also automatically solved. Section 4 is

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<sup>\*)</sup> For details, see for example Ref. 1); any article discusses more or less the physical roles of the partition functions.

concerned with the case where the spinor fields contribute. The presentation is quite parallel to the one in § 3, but this time the Kronecker second limit formula proves to be most relevant. Some additional remarks are contained in § 5. Proofs of some important formulae are collected in the Appendices.

## § 2. A characterization of one-loop domains

Kac once delivered an amusing lecture entitled “Can One Hear the Shape of a Drum?”<sup>4)</sup> in which he exhibited an interesting relation between a geometry of planar domains and the eigen-frequencies associated with the diffusion equation. Let  $D$  be a domain in the  $(s, t)$ -plane and  $\partial D$  its boundary. We then suppose given a function  $\varphi(s, t; T)$ , depending also on the time  $T$ , which satisfies\*) 1)  $\partial\varphi/\partial T = (\partial_s^2 + \partial_t^2)\varphi$ , 2)  $\varphi(s, t; 0) = y(s, t)$  for  $(s, t) \in D$  and 3)  $\varphi(s, t; T) = 0$  for  $(s, t) \in \partial D$  and  $0 < T < \infty$ . Then  $\varphi(s, t; T)$  is written as

$$\varphi(s, t; T) = \int_D ds' dt' y(s', t') P(s, t | s', t'; T), \quad (2.1)$$

where

$$P(s, t | s', t'; T) = \sum_{n=1}^{\infty} e^{-\lambda_n T} \varphi_n(s, t) \varphi_n(s', t'). \quad (2.2)$$

Here  $\{\varphi_n(s, t)\}$  denotes an orthonormal basis for  $L_2(D)$ , each of its components obeying

$$\left. \begin{array}{l} \text{a)} \quad (\partial_s^2 + \partial_t^2) \varphi_n(s, t) + \lambda_n \varphi_n(s, t) = 0, \\ \text{b)} \quad \varphi_n(s, t) = 0 \quad \text{for } (s, t) \in \partial D. \end{array} \right\} \quad (2.3)$$

To see that (2.1) satisfies 1), 2) and 3), it only suffices to try to expand

$$y(s, t) = \sum_{n=1}^{\infty} c_n \varphi_n(s, t). \quad (2.4)$$

The kernel  $P(s, t | s', t'; T)$ , subjected also to the diffusion equation, behaves like  $\delta(s-s')\delta(t-t')$  when  $T \rightarrow 0^+$ , and vanishes for  $(s', t') \in \partial D$ , which thus indicates that the “stuff” is concentrated on  $(s, t)$  at  $T=0$ , while it becomes destroyed as it reaches the boundary. For  $T$  extremely small, the stuff must feel the boundary situate at an infinite distance; hence the approximation for  $T \rightarrow 0^+$ :

$$P(s, t | s', t'; T) \sim \frac{1}{4\pi T} \exp\left[-\frac{(s-s')^2 + (t-t')^2}{4T}\right]. \quad (2.5)$$

Let us tentatively write  $\Theta(t)$  for the quantity

$$\int_D ds dt P(s, t | s, t; T) = \sum_{n=1}^{\infty} e^{-\lambda_n T}, \quad (2.6)$$

which is a trace of a sort. Then it follows from (2.5) that

\*) We here follow a summary by Hejhal.<sup>9)</sup>

$$\Theta(T) \sim \frac{A}{4\pi T} \quad (2.7)$$

for  $T \rightarrow 0^+$ . Here  $A$  denotes the area of  $D$ .\*) According to Kac,<sup>4)</sup>  $\Theta(T)$  more finely expands as

$$\Theta(T) \sim \sum_{n=-2}^{\infty} B_n T^{n/2} \quad (2.7')$$

with

$$B_{-2} = A/4\pi, \quad B_{-1} = L/8\sqrt{\pi}, \quad B_0 = (1-h)/6, \quad \dots \quad (2.8)$$

Here  $L$  stands for the length of  $\partial D$  and  $h$  the number of holes in  $D$ ,  $\dots$ . Hence we may say that if we think of  $D$  as a drum, we can hear and tell first the area of the drum, and then its boundary's length, and thirdly its connectivity,  $\dots$  as time grows. This idea was largely generalized and proved by McKean and Singer.<sup>7)</sup> The coefficient  $B_0$  is related to the Euler characteristic and the like in cases of more generalized manifolds and sometimes is called the index.

It should further be recalled that the Mellin transformation of the *theta-series*  $\Theta(T)$  gives rise to a sort of *zeta-function*:

$$\frac{1}{\Gamma(k)} \int_0^\infty dT T^{k-1} \Theta(T) = \sum_{n=1}^\infty \lambda_n^{-k}, \quad (2.9)$$

which is originally defined for  $\text{Re } k > 1$ , but is analytically continued all over the complex  $k$ -plane except at  $k=1$ . To see that (2.9) has really a simple pole at  $k=1$  with residue  $A/4\pi$ , it will be judicious to apply (2.7) to (2.9) and refer to the fact that  $T_+^{k-2}$  has simple poles at  $k=1, 0, -1, -2, \dots$  (the poles except at  $k=1$  then being cancelled by those of  $\Gamma(k)$ ). This consideration suggests in turn that the first constant term of the Taylor expansion of (2.9) around  $k=0$  is determined by the  $B_0$  term of (2.7'). In fact, if  $B_0=0$ , then the zeta-function (2.9) vanishes at  $k=0$ . It may thus be said that the geometrical information embodied in (2.7') is equivalently contained in the values of the zeta-function. In any case, this field of arithmetical functions is one well established ever since the last century and there is a pile of useful formulae, whose blind applications must sometimes help us to save time in our calculation.

The dual partition functions of mass spectra are constructed as functional integrals on an annulus domain. Our interest rests on the fact that we do not hear the constant term of  $\Theta(T)$ , or the index  $B_0$  vanishes, in the cases of domains with one hole. That is, we want to show how this fact really allows us to evade the difficulty around the problem of the infinite dimensional Jacobian which otherwise bothers us.

\*) Since small  $T$  is large  $\lambda_n$ , Eq. (2.7) alternatively says that the number of  $\lambda_n$  between  $0 < \lambda_n < x$  grows like  $(A/4\pi)x$  as  $x \rightarrow \infty$  (Weyl, 1912—see Ref. 6)).

### § 3. Dual partition function associated with the Veneziano model

#### 3.1. Functional-integral representation

If we tentatively write  $H$  for the Hamiltonian of the dual model, the partition function is roughly given by  $\text{Tre}^{-\beta H}$ , where  $\beta$  is the (rotated) proper time. The dual Hamiltonian is a double integral of the density over, for example, a rectangular domain in the  $(s, t)$ -plane. Hence the trace  $\text{Tr}$  in this case implies a sewing-up of a pair of the opposite sides of the domain. An annulus thus appears on the stage.

Let us define the domain  $C$  by

$$C = \{(s, t) | 0 < s < l\} / \sim, \quad (3.1)$$

where  $\sim$  represents the equivalence relation defined by  $(s, t) \sim (s, 2\beta + t)$ . Let us denote by  $y^{(i)}(s, t)$ ,  $i=1, 2, \dots, \delta$ , the transverse dynamical variables subjected to

$$y^{(i)}(s, t) = 0 \quad \text{for } (s, t) \in \partial C. \quad (3.2)$$

Then the partition function of the orbital string reads

$$A_V(\tau) = \int \mathcal{D}^{\delta} y(s, t) \exp \left\{ -\frac{1}{4\pi} \int_C ds dt [(\partial_s y)^2 + (\partial_t y)^2] \right\}, \quad (3.3)$$

where  $(\partial y)^2 = \sum_{i=1}^{\delta} (\partial y^{(i)})^2$ , and  $\int \mathcal{D}^{\delta} y(s, t) \cdots$  implies the functional integration over all  $y(s, t)$  satisfying (3.2). As a parameter of the partition function, we make constant use of  $\tau$  defined by

$$\tau = i\beta/l. \quad (3.4)$$

In the light of (3.2), we can choose the following as the orthonormal basis corresponding to  $\{\varphi_n\}$  satisfying (2.3):

$$\left. \begin{aligned} \varphi_{\nu}(s, t) &= \sqrt{1/\beta l} \sin \nu(\pi/l)t, \\ \varphi_{\mu\nu}^1(s, t) &= \sqrt{2/\beta l} \cos \mu(\pi/\beta)s \cdot \sin \nu(\pi/l)t, \\ \varphi_{\mu\nu}^2(s, t) &= \sqrt{2/\beta l} \sin \mu(\pi/\beta)s \cdot \sin \nu(\pi/l)t, \end{aligned} \right\} \quad (\mu, \nu = 1, 2, \dots) \quad (3.5)$$

with eigenvalues

$$\left. \begin{aligned} \lambda_{\nu} &= (\nu/l)^2 \cdot \pi^2 & \text{for } \varphi_{\nu}(s, t), \\ \lambda_{\mu\nu} &= [(\mu/\beta)^2 + (\nu/l)^2] \cdot \pi^2 & \text{for } \varphi_{\mu\nu}^i(s, t). \end{aligned} \right\} \quad (3.6)$$

Then, corresponding to (2.4), we have the expansion

$$\begin{aligned} y(s, t) &= \sum_{\nu=1}^{\infty} a_{\nu} \varphi_{\nu}(s, t) \\ &\quad + \sum_{\mu=1}^{\infty} \sum_{\nu=1}^{\infty} [a_{\mu\nu}^1 \varphi_{\mu\nu}^1(s, t) + a_{\mu\nu}^2 \varphi_{\mu\nu}^2(s, t)], \end{aligned} \quad (3.7)$$

where the coefficients  $(a_{\nu}, a_{\mu\nu}^1, a_{\mu\nu}^2)$  are  $\delta$ -vectors.

Let us regard (3.7) as usual as defining the variable transformation from  $y(s, t)$  for  $(s, t) \in C$  to  $(a_\nu, a_{\mu\nu}^1, a_{\mu\nu}^2)$  for  $\mu, \nu = 1, 2, \dots$ . Its Jacobian is necessarily indefinite. We therefore scale  $y(s, t)$  to  $jy(s, t)$  with some constant  $j$  and assume we can fix  $j$  in such a way that the measure remains the same if the Jacobian from  $y(s, t)$  to  $(ja_\nu, ja_{\mu\nu}^1, ja_{\mu\nu}^2)$  is unity (this is always possible whenever the dimension is finite). This then allows us to write the r.h.s. of (3.3) as

$$\begin{aligned} A_V(\tau) &= \int \prod_{\nu=1}^{\infty} j d^\delta a_\nu \prod_{\mu, \nu=1}^{\infty} j^2 d^\delta a_{\mu\nu}^1 d^\delta a_{\mu\nu}^2 \\ &\quad \times \exp \left\{ -\frac{1}{4\pi} \lambda_\nu (a_\nu)^2 - \frac{1}{4\pi} \lambda_{\mu\nu} [(a_{\mu\nu}^1)^2 + (a_{\mu\nu}^2)^2] \right\} \\ &= \prod_{\nu=1}^{\infty} j^\delta (4\pi^2)^{\delta/2} \prod_{\mu, \nu=1}^{\infty} j^{2\delta} (4\pi^2)^\delta \prod_{\nu=1}^{\infty} \lambda_\nu^{-\delta/2} \prod_{\mu, \nu=1}^{\infty} \lambda_{\mu\nu}^{-\delta}. \end{aligned} \quad (3.8)$$

The last line clearly suggests that we should define the zeta-function as follows:

$$Z_a(k) = \frac{1}{2} \sum_{\nu=1}^{\infty} \frac{1}{\lambda_\nu^k} + \sum_{\mu, \nu=1}^{\infty} \frac{1}{\lambda_{\mu\nu}^k} \quad (3.9)$$

corresponding to (2.9). In fact, then, the r.h.s. of (3.8), as exemplified in Hawking's article,<sup>5)</sup> takes the following compact form:

$$A_V(\tau) = (2\pi j)^{2\delta Z_a(0)} \exp[\delta \cdot dZ_a(k)/dk|_{k=0}]. \quad (3.10)$$

One of the cruxes in the previous expression (3.8) was that infinite products of  $2\pi j$  appeared in it. For the reasons set out in § 2, however,  $Z_a(k)$  is expected to vanish at  $k=0$  in the present case so that  $A_V(\tau)$  may simply prove to be

$$A_V(\tau) = \exp[\delta \cdot Z_a'(0)]. \quad (3.10')$$

As is widely known, this expression, if straightforwardly calculated, still suffers from a divergent factor due to the presence of the “zero-point energies”. The value of our method in what follows also lies in an automatic resolution of this trouble.

It is easily inspected that  $A_V(\tau)$  does not respect by construction the invariance under the interchange of  $l$  and  $\beta$  (or  $\tau \rightarrow -1/\tau$ ); but it will be proved that  $A_V(\tau)$  is a modular form on  $SL(2, Z)$  with respect to  $\tau$ .

### 3.2. Evaluation

The first term of  $Z_a(k)$  of (3.9) is a standard Riemann zeta-function and the second belongs to Epstein's zeta-function.<sup>8)</sup> As a typical case of the Epstein zeta-function the following Dirichlet series is often used<sup>8)~10)</sup>

$$E(\tau, k) = \sum_{(\mu, \nu) \neq (0, 0)} \frac{(\text{Im } \tau)^k}{|\nu\tau + \mu|^{2k}}, \quad (3.11)$$

in which  $(\mu, \nu)$  runs over the lattice which consists of all pairs of integers except for  $(0, 0)$ . Hence  $Z_a(k)$  is alternatively written as

$$Z_a(k) = \frac{1}{4} (l\beta/\pi^2)^k E(i\beta/l, k) - \frac{1}{2} (\beta/\pi)^{2k} \zeta(2k). \quad (3.12)$$

One can find one integral representation for  $E(\tau, k)$  in Ref. 8) and another in Ref. 11), from which one may readily obtain the values of  $Z_a(k)$ . However the values we need are only at a few points so that we can proceed differently.

As is recaptured in Lang [Ref. 10), Chap. 20, § 4],  $E(\tau, k)$  has a simple pole at  $k=1$  and satisfies

$$\left. \begin{aligned} \lim_{k \rightarrow 1} (k-1) E(\tau, k) &= \pi, \\ \lim_{k \rightarrow 1} \frac{d}{dk} (k-1) E(\tau, k) &= -\pi \log |\tau| + 2\pi(\gamma - \log 2) - 4\pi \log |\eta(\tau)|, \end{aligned} \right\} \quad (3.13)$$

where  $\gamma$  is the Euler constant and  $\eta(\tau)$  the Dedekind eta-function

$$\eta(\tau) = e^{\pi i \tau / 12} \prod_{\nu=1}^{\infty} (1 - e^{2\nu \pi i \tau}). \quad (3.14)$$

The first of (3.13) has essentially the same content as (2.7), and the second formula of (3.13) is the one called *the Kronecker first limit formula*.

The material in Ref. 10) is well sufficient to show that  $E(\tau, k)$  obeys the following *functional equation*:

$$\pi^{-k} \Gamma(k) E(\tau, k) = \pi^{k-1} \Gamma(1-k) E(\tau, 1-k). \quad (3.15)$$

This and the first formula of (3.13) suffice to prove explicitly

$$Z_a(0) = 0. \quad (3.16)$$

In fact, it follows first from (3.15) that

$$\begin{aligned} E(\tau, 0) &= \lim_{k \rightarrow 0} \pi^{-k} \Gamma(1+k) E(\tau, k) \\ &= \pi^{-1} \Gamma(1) \lim_{k \rightarrow 0} k E(\tau, 1-k), \end{aligned} \quad (3.17)$$

which is  $-1$  however due to (3.13). On the other hand, we have  $\zeta(0) = -1/2$ ; hence the assertion (3.16).

The calculation of  $Z_a'(0)$  is also straightforward: As shown in Appendix A, we firstly have

$$\left. \frac{d}{dk} E(\tau, k) \right|_{k=0} = \pi^{-1} \left. \frac{d}{dk} (k-1) E(\tau, k) \right|_{k=1} - 2 \log \pi - 2\gamma, \quad (3.18)$$

the r.h.s. of which is written as

$$-\log |\tau| \cdot |\eta(\tau)|^4 (2\pi)^2$$

due to (3.13). On the other hand, we have  $\zeta'(0) = -(1/2) \log 2\pi$  and  $\zeta(0) = -1/2$ . Hence

$$\begin{aligned} Z_a'(0) &= -\frac{1}{4} \log(l\beta/\pi^2) + \frac{1}{4} \left. \frac{d}{dk} E(\tau, k) \right|_{k=0} - \zeta'(0) - \zeta(0) \cdot \log(\beta/\pi) \\ &= -\log |\eta(\tau)|. \end{aligned} \quad (3.19)$$

Upon inserting this into (3·10'), we reach the final result:

$$A_V(\tau) = \eta(\tau)^{-\delta}. \quad (3 \cdot 20)$$

The function  $\eta(\tau)$  is a typical modular form on the modular group generated by  $\tau \rightarrow \tau + 1$  and  $\tau \rightarrow -1/\tau$ ,<sup>\*)</sup> and so is  $A_V(\tau)$ . Since  $\eta(\tau)$  has a zero at the cusp  $\tau \rightarrow i\infty$ ,  $A_V(\tau)$  has a pole at  $\tau \rightarrow i\infty$ . If this is a pole with multiplicity *one*, then  $\delta = 24$  (the local variable at  $\tau \rightarrow i\infty$  is  $\exp(2\pi i\tau)$ ), and the pole is the tachyon pole.

We close this section by remarking that any constant multiple of the exponent in (3·3) (for example  $1/4\pi \rightarrow 1/4\pi\alpha'$ ) is however indifferent to the final result (3·20).

#### § 4. Contributions from spinor variables

##### 4.1. Functional-integral representation

In place of  $y^{(i)}(s, t)$ , we now take account of two conformal spinor  $\delta$ -fields  $\phi_1^{(i)}(s, t)$  and  $\phi_2^{(i)}(s, t)$  in  $C$  along the line of Refs. 13) and 14). The boundary condition is given by

$$\phi_1(s, t)\phi_2(s, t) = 0 \quad \text{at} \quad \partial C. \quad (4 \cdot 1)$$

Out of the contents of (4·1), we here take out the case which Virasoro<sup>14)</sup> firstly chose, that is, the case where

$$\left. \begin{aligned} \bar{\phi}_1 &= \phi_1 = 0 & \text{at } s=l, \\ \bar{\phi}_2 &= \phi_2 = 0 & \text{at } s=0. \end{aligned} \right\} \quad (4 \cdot 2)$$

Then we have the expansions

$$\left. \begin{aligned} \phi_1(s, t) &= \sum_{\nu=1}^{\infty} B_{\nu}^1(t) \cos\left(\nu - \frac{1}{2}\right) \frac{\pi}{l} s, \\ \phi_2(s, t) &= \sum_{\nu=1}^{\infty} B_{\nu}^2(t) \sin\left(\nu - \frac{1}{2}\right) \frac{\pi}{l} s. \end{aligned} \right\} \quad (4 \cdot 3)$$

We next invoke periodic conditions

$$\phi_i(s, t + 2m\beta) = -\phi_i(s, t), \quad (i=1, 2) \quad (4 \cdot 4)$$

the minus sign corresponding to the even  $G$ -parity condition of the Neveu-Schwarz model.<sup>2)</sup> Here  $m$  denotes a rational number (usually  $m=1$ ). Then we further have

$$\left. \begin{aligned} B_{\nu}^1(t) &= \sqrt{\frac{2}{m\beta l}} \sum_{\mu=1}^{\infty} \left[ b_{\mu\nu}^{11} \sin\left(\mu - \frac{1}{2}\right) \frac{\pi}{m\beta} t + b_{\mu\nu}^{12} \cos\left(\mu - \frac{1}{2}\right) \frac{\pi}{m\beta} t \right], \\ B_{\nu}^2(t) &= \sqrt{\frac{2}{m\beta l}} \sum_{\mu=1}^{\infty} \left[ b_{\mu\nu}^{21} \sin\left(\mu - \frac{1}{2}\right) \frac{\pi}{m\beta} t + b_{\mu\nu}^{22} \cos\left(\mu - \frac{1}{2}\right) \frac{\pi}{m\beta} t \right], \end{aligned} \right\} \quad (4 \cdot 5)$$

<sup>\*)</sup> As to the theory of modular groups, there is a nice review by Yui for theoretical physicists.<sup>12)</sup>

in which  $b_{\mu\nu}^{ij}$  are anti-commuting  $c$ -numbers:

$$\{b_{\mu\nu}^{ij}, b_{\mu'\nu'}^{i'j'}\} = 0. \quad (4.6)$$

The contribution to the partition function in a functional-integral form is then given by

$$A_{b,m}(\tau) = \int \mathcal{D}^s \psi_1(s, t) \mathcal{D}^s \psi_2(s, t) \cdot \exp \left\{ -\frac{1}{\pi} \int_C ds dt (\psi_1, \psi_2) \begin{pmatrix} \vec{\partial}_t & \vec{\partial}_s \\ \vec{\partial}_s & -\vec{\partial}_t \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \right\}, \quad (4.7)$$

where the functional integral should be performed all over the  $\psi_1$  and  $\psi_2$  satisfying (4.1).

Let us change the variables as before from  $\{\psi_1, \psi_2(s, t)\}$  to  $\{b_{\mu\nu}^{ij}\}$  in such a way that the Jacobian times  $\prod_{\mu,\nu=1}^{\infty} \prod_{i,j} d^s b_{\mu\nu}^{ij}$  becomes identical to  $\prod d^s (j_b b_{\mu\nu}^{ij})$  with some scaling constant  $j_b$ . Then the r.h.s. of (4.7) takes the form

$$A_{b,m}(\tau) = \int \prod_{\mu,\nu} \prod_{i,j} j_b d^s b_{\mu\nu}^{ij} \exp \left\{ \frac{1}{2} \sum_{\mu,\nu} \varphi_{\mu\nu} M_{\mu\nu} \varphi_{\mu\nu} \right\}, \quad (4.8)$$

where

$$\varphi_{\mu\nu} = (b_{\mu\nu}^{11}, b_{\mu\nu}^{12}, b_{\mu\nu}^{21}, b_{\mu\nu}^{22}) \quad (4.9)$$

and

$$M_{\mu\nu} = 4 \begin{pmatrix} 0 & (\mu - \frac{1}{2})/m\beta & -(\nu - \frac{1}{2})/l & 0 \\ -(\mu - \frac{1}{2})/m\beta & 0 & 0 & -(\nu - \frac{1}{2})/l \\ (\nu - \frac{1}{2})/l & 0 & 0 & -(\mu - \frac{1}{2})/m\beta \\ 0 & (\nu - \frac{1}{2})/l & (\mu - \frac{1}{2})/m\beta & 0 \end{pmatrix}. \quad (4.10)$$

The integral (4.8) is a Gaussian-type one on a Grassmann algebra generated by  $b_{\mu\nu}^{ij}$ , and the way to reach the following result is well known.<sup>15), 16)</sup>

$$A_{b,m}(\tau) = \prod_{\mu,\nu=1}^{\infty} j_b^{4s} (\det M_{\mu\nu})^{\delta/2} \quad (4.11)$$

with

$$(\det M_{\mu\nu})^{1/2} = 2 \left[ \left( \frac{1}{m\beta} \right)^2 \left( \mu - \frac{1}{2} \right)^2 + \left( \frac{1}{l} \right)^2 \left( \nu - \frac{1}{2} \right)^2 \right]. \quad (4.12)$$

As companions of (3.6) and (3.9), let us formally define

$$\lambda'_{\mu\nu} = \left[ \left( \frac{\pi}{m\beta} \right)^2 \left( \mu - \frac{1}{2} \right)^2 + \left( \frac{\pi}{l} \right)^2 \left( \nu - \frac{1}{2} \right)^2 \right] \quad (4.13)$$

and

$$Z_{\delta,m}(k) = \sum_{\mu,\nu=1}^{\infty} (\lambda'_{\mu\nu})^{-k}. \quad (4.14)$$

Then the r.h.s. of (4.11) turns out to be



$$A_{b,m}(\tau) = (2j_b^4/\pi^2)^{\delta Z_{b,m}(0)} \exp[-\delta \cdot Z'_{b,m}(0)]. \quad (4.15)$$

#### 4.2. Evaluation

Let us define a new Dirichlet series:

$$F_b(\tau, k) = \sum_{\mu, \nu=1}^{\infty} \frac{(\operatorname{Im} \tau)^k}{|(\nu - \frac{1}{2})\tau + (\mu - \frac{1}{2})|^{2k}}. \quad (4.16)$$

Then

$$Z_{b,m}(k) = \left(\frac{m\beta l}{\pi^2}\right)^k F_b(m\tau, k). \quad (4.17)$$

Let us further invoke

$$E_{1/2, 1/2}(\tau, k) = \sum_{(\mu, \nu) \neq (0, 0)} \frac{(-)^{\mu+\nu} (\operatorname{Im} \tau)^k}{|\nu\tau + \mu|^{2k}}, \quad (4.18)$$

where the sum is taken over all pairs of integers except for  $(0, 0)$ . Then, as shown in Appendix B,  $F_b(m\tau, k)$  is paraphrased as

$$\begin{aligned} \Gamma(k) F_b(m\tau, k) \\ = \frac{1}{4} \pi^{2k-1} \Gamma(1-k) E_{1/2, 1/2}(m\tau, 1-k). \end{aligned} \quad (4.19)$$

The series (4.18) belongs to

$$E_{u,v}(\tau, k) = \sum_{(\mu, \nu) \neq (0, 0)} e^{2\pi i(\nu u + \mu v)} \frac{(\operatorname{Im} \tau)^k}{|\nu\tau + \mu|^{2k}} \quad (4.20)$$

whose definition may be traced back for example to Ref. 8) and is also found in Ref. 10), p. 276. Its marked feature is that it, when  $u, v$  are not both integers, can be analytically continued to  $k=1$  and becomes entire. It follows then that the r.h.s. of (4.19) is analytic at any neighbourhood of  $k=0$ , and so is the l.h.s. On the other hand  $\Gamma(k)$  has a pole at  $k=0$  so that  $F_b(\tau m, k)$  should vanish at the very point. Hence we have proved

$$Z_{b,m}(0) = 0. \quad (4.21)$$

The value of the derivative of  $Z_{b,m}(k)$  at  $k=0$  is also straightforwardly obtainable: Let us multiply both sides of (4.19) by  $k$ , differentiate and put them to the limit  $k=0$ . Then we readily have

$$\left. \frac{d}{dk} F_b(m\tau, k) \right|_{k=0} = \frac{1}{4\pi} E_{1/2, 1/2}(m\tau, 1). \quad (4.22)$$

On the other hand, Kronecker's second limit formula<sup>10)</sup> tells us

$$E_{1/2, 1/2}(m\tau, 1) = -4\pi \log |e^{-\pi i m\tau/24} \prod_{\nu=1}^{\infty} (1 + e^{(2\nu-1)\pi i m\tau})|. \quad (4.23)$$

Equations (4.22) and (4.21) suffice to obtain the value of  $Z'_{b,m}(0)$ , and upon inserting this result as well as (4.21) into (4.15), we eventually obtain

$$A_{b,m}(\tau) = e^{-\pi i m \tau \delta / 24} \prod_{\nu=1}^{\infty} (1 + e^{(2\nu-1)\pi i m \tau})^{\delta}. \quad (4.24)$$

In terms of Jacobi's theta functions, the r.h.s. of (4.24) for  $m=1$  is written as

$$A_{b,1}(\tau) = (2\vartheta_3^2(0|\tau) / \vartheta_2(0|\tau) \vartheta_4(0|\tau))^{\delta/3}. \quad (4.25)$$

which clearly shows that  $A_{b,1}(\tau)$  is automorphic under  $\tau \rightarrow -1/\tau$  and  $\tau \rightarrow \tau + 2$ .

### § 5. Concluding remarks

1° The approach we have adopted thus far has a merit of nowhere suffering from the divergence trouble when producing the tachyon factor  $e^{-\pi i \tau \delta / 12}$  in (3.20) or  $e^{-\pi i \tau \delta / 24}$  in (4.24): This is thanks to the functional equation which governs  $E(\tau, k)$  or  $E_{1/2, 1/2}(\tau, k)$ . It is also simply reasoned that the negative mass-squared of tachyon arises from an infinite series of “zero-point energies” in response to  $\zeta(-1) = 1 + 2 + 3 + \dots = -1/12$ . (See for example Ref. 17).)

2° The tachyon factors are also known to determine the effective space-time dimensions. Let us now consider the partition function of the Neveu-Schwarz string: It is given by

$$\begin{aligned} A_{NS}(\tau) &= A_V(\tau) A_{b,1}(\tau) \\ &= e^{-\pi i \tau \delta / 8} \prod_{\nu=1}^{\infty} (1 - e^{2\nu \pi i \tau})^{-\delta} (1 + e^{(2\nu-1)\pi i \tau})^{\delta}. \end{aligned} \quad (5.1)$$

Taking account of  $\eta(\tau) = (\frac{1}{2} \vartheta_2(0|\tau) \vartheta_3(0|\tau) \vartheta_4(0|\tau))^{1/3}$  and (4.24), we also write it as

$$A_{NS}(\tau) = [\frac{1}{2} \vartheta_2(0|\tau) \vartheta_4(0|\tau)]^{-\delta/2} \quad (5.1')$$

the r.h.s. of which clearly informs us that  $A_{NS}(\tau)$  itself is a modular form with weight  $-\delta/2$  on the subgroup  $\Gamma_\theta$  generated by  $\tau \rightarrow \tau + 2$  and  $\tau \rightarrow -1/\tau$ .<sup>18)</sup> Because, for the case where the group is generated by  $\tau \rightarrow \tau + 2$ , the local variable at  $\tau \rightarrow i\infty$  is given by  $e^{\pi i \tau}$ , the first factor in (5.1) readily tells us that  $\delta$  should be 8 if the pole is a pole with multiplicity one.

3° In § 4, we inserted a rational number  $m$  as a free parameter. Let us here try to take out the cases of  $m=1/2$  and  $m=2$  in addition to that of  $m=1$ . The effective dimensions may also be different in respective cases. We however assume the dimensions for  $m=1/2$  and  $m=2$  are identical and denote it by  $\delta'$ . Then

$$A_{b,1/2}(\tau) A_{b,2}(\tau) = \left( \frac{4\vartheta_3^2(0|2\tau) \vartheta_3^2(0|\tau/2)}{\vartheta_2(0|2\tau) \vartheta_4(0|\tau/2) \vartheta_2(0|\tau/2) \vartheta_4(0|2\tau)} \right)^{\delta'/8} \quad (5.2)$$

and the r.h.s. proves to be automorphic under  $\mathfrak{G}(4)$  generated by  $\tau \rightarrow \tau + 4$  and  $\tau \rightarrow -1/\tau$ . Hence so is  $A_V(\tau) A_{b,1}(\tau) A_{b,1/2}(\tau) A_{b,2}(\tau)$ , whose tachyon factor is given by

$$\exp \left\{ -i\pi\tau \left( \frac{\delta}{12} + \frac{\delta_1}{24} + \frac{5}{48} \delta' \right) \right\}, \quad (5.3)$$

where  $\delta_1$  denotes the effective dimension for the case  $m=1$ . Since the local variable at  $\tau \rightarrow i\infty$  in this case is given by  $e^{i\pi\tau/2}$ , we have the relation

$$4\delta + 2\delta_1 + 5\delta' = 24 \quad (5.4)$$

with the proviso that the multiplicity of the tachyon pole is one. It follows from (5.4) that  $\delta' = 2$ , and so the possible configurations of  $(\delta, \delta_1)$  are given by (1, 5), (3, 1) and (2, 3). With the choice of the last case ( $\delta = 2!$ ),  $A_V(\tau) A_{b,1}(\tau) A_{b,1/2}(\tau) \times A_{b,2}(\tau)$  is nothing but the partition function  $A_N(\tau)$  extensively dealt with in Ref. 19). This is a modular form of weight  $-\delta/2 (= -1)$  on the group  $\Gamma_N$  which is somewhat larger than  $\mathfrak{S}(4)$ .

4° The partition function of the Ramond string is also interesting and obtainable by the same kind of tools used in § 3: Let us here repeat part of the arguments. It is the following choice of the implications of the boundary condition (4.1) that distinguishes the Ramond model from the Neveu-Schwarz model:<sup>14)</sup>

$$\bar{\psi}_2 = \psi_2 = 0 \text{ both at } s=0 \text{ and } s=l \quad (5.5)$$

(cf. (4.2)). By virtue of this, we have

$$\left. \begin{aligned} \psi_1(s, t) &= \sum_{\nu=0}^{\infty} D_{\nu}^1(t) \cos \nu \frac{\pi}{l} s, \\ \psi_2(s, t) &= \sum_{\nu=1}^{\infty} D_{\nu}^2(t) \sin \nu \frac{\pi}{l} s \end{aligned} \right\} \quad (5.6)$$

instead of (4.3), while the  $D_{\nu}^i(t)$ 's are subjected to the expansions similar to  $B_{\nu}^i(t)$  (see (4.5)). It should be recalled that  $\psi_1(s, t)$  has a zeroth mode term, and hence we have eventually to define the zeta-function as

$$Z_d(k) = \frac{1}{2} \left( \frac{\beta}{\pi} \right)^{2k} \sum_{\mu=1}^{\infty} \frac{1}{(\mu - \frac{1}{2})^{2k}} + \left( \frac{l\beta}{\pi^2} \right) \sum_{\mu, \nu=1}^{\infty} \frac{|\tau|^k}{|\nu\tau + \mu - \frac{1}{2}|^{2k}}, \quad (5.7)$$

the first series of which is proportional to the Hurwitz zeta-function  $\zeta(2k, \frac{1}{2}) [\zeta(0, \frac{1}{2}) = 0!]$ . The second series is related to  $E_{1/2,0}(\tau, k)$  (a specific case of (4.20)), whose value at  $k=1$  is given by<sup>10)</sup>

$$E_{1/2,0}(\tau, 1) = -2\pi \log 2 - 4\pi \log |e^{i\pi\tau/12} \prod_{\nu=1}^{\infty} (1 + e^{2\nu i\pi\tau})|. \quad (5.8)$$

The  $\log 2$  term is to be cancelled by the term containing  $\zeta'(0, \frac{1}{2}) = -\frac{1}{2} \log 2$ , and we are finally led to

$$A_d(\tau) = e^{i\pi\tau\delta/12} \prod_{\nu=1}^{\infty} (1 + e^{2\nu i\pi\tau})^{\delta} \quad (5.9)$$

or to

$$A_R(\tau) = A_V(\tau) A_d(\tau) = \prod_{\nu=1}^{\infty} (1 - e^{2\nu i\pi\tau})^{-\delta} (1 + e^{2\nu i\pi\tau})^{\delta}. \quad (5.10)$$

In addition to  $E_{1/2,1/2}(\tau, 1)$  and  $E_{1/2,0}(\tau, 1)$ , we may naturally feel concern for the

presence of  $E_{0\ 1/2}(\tau, 1)$ : These really satisfy

$$\exp\left[-\frac{2}{\pi}E_{1/2\ 1/2}(\tau, 1)\right] = \exp\left[-\frac{2}{\pi}E_{1/2\ 0}(\tau, 1)\right] + \exp\left[-\frac{2}{\pi}E_{0\ 1/2}(\tau, 1)\right] \quad (5.11)$$

corresponding to Jacobi's *aequatio identica satis abstrusa*.<sup>20)</sup>

5° In case we try to establish a dual-resonance amplitude in the framework of the functional approach, we must calculate a functional integral similar to (3.3) to yield its volume element on the rectangular domain

$$\{(s, t) | 0 < s < l, 0 < t < \beta\}. \quad (5.12)$$

In this case, according to McKean and Singer,<sup>7)</sup> the index  $B_0$  does not vanish but is  $1/4$ . Hence the zeta-function is forced to differ from zero at the origin, and it becomes needed to differently regularize the infinite dimensional Jacobian. This will perhaps be performed by a presence of a counter behaviour of the integrand, which is however another matter, not well founded.

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### Appendix A

We here give a proof of (3.18): Let us multiply the functional equation (3.15) by  $k-1$ . Then

$$\pi^{-k}\Gamma(k) \cdot (k-1)E(\tau, k) = -\pi^{k-1}\Gamma(2-k) \cdot E(\tau, 1-k). \quad (A.1)$$

Let us differentiate both the sides and put  $k=1$ . Then

$$\begin{aligned} & (\pi^{-k}\Gamma(k))' \Big|_{k=1} \cdot \pi + \pi^{-1} \left( \frac{d}{dk} (k-1)E(\tau, k) \right)' \Big|_{k=1} \\ &= -(\pi^{k-1}\Gamma(2-k))' \Big|_{k=1} E(\tau, 0) - \Gamma(1) \cdot \left( \frac{d}{dk} E(\tau, 1-k) \right)' \Big|_{k=1} \end{aligned} \quad (A.2)$$

or

$$\begin{aligned} & \frac{d}{dk} E(\tau, k) \Big|_{k=0} - \pi^{-1} \left( \frac{d}{dk} (k-1)E(\tau, k) \right)' \Big|_{k=1} \\ &= \pi(\pi^{-k}\Gamma(k))' \Big|_{k=1} + (\pi^{k-1}\Gamma(2-k))' \Big|_{k=1}, \end{aligned} \quad (A.3)$$

in which we have applied  $E(\tau, 0) = -1$ , (3.17). The r.h.s. of (A.3) is just  $-2\log \pi + 2\psi(k)\Gamma(k) \Big|_{k=1}$  where  $\psi(k) = \Gamma'(k)/\Gamma(k)$ . Since  $\psi(1) = -\gamma$ , Eq. (3.18) is thus verified.

### Appendix B

We next derive here the functional equation (4.19) between  $F_b(\tau, k)$  and  $E_{1/2, 1/2}(\tau, k)$  given by (4.16) and (4.18) respectively. By virtue of

$$\pi^{-k} \Gamma(k) a^{-k} = \int_0^\infty dt t^{k-1} e^{-\pi a t}, \quad (\text{B} \cdot 1)$$

we readily have

$$\begin{aligned} I_1(k) &:= 4\pi^{-k} \Gamma(k) |\tau|^{k-1} F_b(\tau, k) \\ &= 4|\tau|^{2k-1} \sum_{\mu=1}^\infty \sum_{\nu=1}^\infty \int_0^\infty dt t^{k-1} e^{-\pi[(\nu-(1/2))^2|\tau|^2 + (\mu-(1/2))^2]t}, \end{aligned} \quad (\text{B} \cdot 2)$$

where  $\tau = i\beta/l$ , (3.4). Let us apply to (B.2)

$$\sum_{n=1}^\infty e^{-\pi(n-(1/2))^2 a t} = \frac{1}{2\sqrt{at}} \sum_{n=-\infty}^\infty (-)^n e^{-\pi n^2/at}, \quad (\text{B} \cdot 3)$$

which is for instance derived by the Jacobi transformation  $\vartheta_4(0|\tau') = (-i\tau')^{-1/2} \times \vartheta_2(0|-1/\tau')$ . Then

$$\begin{aligned} I_1(k) &= 2|\tau|^{2k-1} \sum_{\nu=1}^\infty \int_0^\infty dt t^{k-(3/2)} e^{-\pi(\nu-(1/2))^2|\tau|^2 t} \\ &\quad + |\tau|^{2k-2} \sum_{\mu=-\infty}^\infty (-)^\mu \int_0^\infty dt t^{k-2} e^{-\pi\mu^2/t} \\ &\quad + |\tau|^{2k-1} \sum_\mu' \sum_\nu' (-)^{\mu+\nu} \int_0^\infty dt t^{k-2} e^{-(\pi\nu^2/t|\tau|^2) - (\pi\mu^2/t)}, \end{aligned} \quad (\text{B} \cdot 4)$$

where  $\sum'$  denotes the summation taken for all integers  $\mu$  or  $\nu \neq 0$ . The first integral of (B.4) is proportional to a Hurwitz zeta-function. The last two integrals admit both the change of integration variable  $ty^2 \rightarrow 1/t$ . Then

$$I_1(k) = J_0(k) + J_1(k) + J_2(k) \quad (\text{B} \cdot 5)$$

with

$$\begin{aligned} J_0(k) &:= 2\pi^{-k+1/2} \Gamma(k-\tfrac{1}{2}) \zeta(2k-1, \tfrac{1}{2}) \\ &= 2\pi^{k-1} \Gamma(1-k) (2^{2k-1}-1) \zeta(2-2k), \end{aligned} \quad (\text{B} \cdot 6)$$

$$J_1(k) := \sum_\nu' (-)^\nu \int_0^\infty dt t^{-k} e^{-\pi\nu^2|\tau|^2 t} \quad (\text{B} \cdot 7)$$

and

$$J_2(k) := \sum_\mu' \sum_\nu' (-)^{\mu+\nu} \int_0^\infty dt t^{-k} e^{-\pi\nu^2|\tau|^2 t - \pi\mu^2/t}. \quad (\text{B} \cdot 8)$$

Now we turn to  $E_{1/2, 1/2}(\tau, k)$ : Application of (B.1) readily yields

$$\begin{aligned} I_2(k) &:= \pi^{-k} \Gamma(k) |\tau|^{-k} E_{1/2, 1/2}(\tau, k) \\ &= 2 \sum_{\mu=1}^\infty (-)^\mu \int_0^\infty dt t^{k-1} e^{-\pi\mu^2 t} + \sum_\nu' (-)^\nu \int_0^\infty dt t^{k-1} e^{-\pi\nu^2|\tau|^2 t} \end{aligned}$$

$$+ \sum_{\mu}' \sum_{\nu}' (-)^{\mu+\nu} \int_0^{\infty} dt t^{k-1} e^{-\pi\nu^2|\tau|^2 - \pi\mu^2 t}. \quad (\text{B}\cdot 9)$$

The first term of (B·9) gives  $2\pi^{-k}\Gamma(k)(2^{1-2k}-1)\zeta(2k)$  and further an inspection tells us that the second is identical to  $J_1(1-k)$  and the third to  $J_2(1-k)$ . We have thus verified

$$I_1(k) = I_2(1-k), \quad (\text{B}\cdot 10)$$

hence (3·19):

$$4\pi^{-k}\Gamma(k)F_b(\tau, k) = \pi^{k-1}\Gamma(1-k)E_{1/2, 1/2}(\tau, 1-k). \quad (\text{B}\cdot 11)$$

The last expression is quite similar to (3·15), but the r.h.s. is not quite the old self; this is perhaps related to the fact that the number of the cusps of the fundamental region of  $\Gamma_\theta$  or its adjoints is not unity.

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