Indefinite-Metric Quantum Field Theory of General Relativity. V
—Vierbein Formalism—

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The indefinite-metric quantum field theory of general relativity is extended to the coupled system of the gravitational field and a Dirac field on the basis of the vierbein formalism. The six extra degrees of freedom involved in vierbein are made unobservable by introducing an extra subsidiary condition $Q,|\text{phys}\rangle=0$, where $Q$, denotes a new BRS charge corresponding to the local Lorentz invariance. It is shown that a manifestly covariant, unitary, canonical theory can be constructed consistently on the basis of the vierbein formalism.

§ 1. Introduction

In a series of papers, we have successfully developed the indefinite-metric quantum field theory of gravity. In the first paper, we emphasized the importance of the following four fundamental principles: Lagrangian and canonical formalism, manifest covariance, indefinite-metric Hilbert space with subsidiary conditions (so as to make the physical $S$-matrix unitary), and asymptotic completeness. The present author believes that the true fundamental theory describing Nature should satisfy the above four principles, unless space-time itself needs to be quantized.

The existence of Dirac fields in Nature is undoubtedly true. In order to conform to our standpoint, therefore, we must extend our indefinite-metric quantum field theory of general relativity to the coupled system of the gravitational field and a Dirac field. This problem is highly non-trivial, because the generally-covariant formulation of a Dirac field cannot be made in terms of the metric tensor $g_{\mu\nu}$ alone. As is well known, the Dirac theory is most conveniently formulated in terms of vierbein. The vierbein $e_{\mu}^a$ ($a=0, 1, 2, 3$) involves six extra degrees of freedom, which are nothing but the freedom of choosing the directions of the four axes labeled as $a=0, 1, 2, 3$ at each space-time point. Since the transformation between two choices of four axes is a Lorentz transformation, it is usually called the local Lorentz (LL) transformation, though this name is somewhat misleading, because it is not a coordinate transformation.

It is known that the generally-covariant Dirac Lagrangian density is invariant under the LL transformation. Accordingly, the new situation encountered here is quite similar to the Yang-Mills theory; we have a local internal symmetry, which
is a Lorentz group. It is natural, therefore, to introduce the BRS transformation corresponding to it and set up a Kugo-Ojima subsidiary condition.\(^6\)

In the present paper, we show that the extension of our canonical formalism\(^7\) to the vierbein case is carried out consistently. In § 2, we review the generally-covariant formulation of a Dirac field in terms of vierbein. In § 3, after defining the LL-BRS transformation, we introduce the LL-gauge-fixing Lagrangian density and the LL-FP-ghost one. Then a new system of field equations is obtained. In § 4, we discuss the LL-FP-ghost current \(J_f\), the LL-BRS current \(J_b\), the FP-ghost current \(J_b\), the BRS current \(J_b\) and the Poincaré generators \(P\) and \(M\). It is shown that the expressions for \(Q_c\), \(Q_b\) and \(P\) remain unchanged. In § 5, we introduce the asymptotic fields and show the unitarity of the physical \(S\)-matrix in the Heisenberg picture. Discussion is made on our choice of the LL-gauge-fixing term in the final section.

The analysis of commutation relations will be presented in a succeeding paper.

\section*{§ 2. Vierbein and a Dirac field}

We denote the vierbein by \(e_{\mu a}\), which satisfy

\[\gamma_{ab} e_{\mu a} e_{\nu b} = g_{\mu \nu}, \quad (2.1)\]

\[\gamma^{ab} e_{\mu a} e_{\nu b} = \eta_{\mu \nu}, \quad (2.2)\]

where \(g_{\mu \nu}\) and \(\eta_{\mu \nu}\) are the gravitational field and the Minkowski metric \((+ - - -)\), respectively. Greek indices and Latin ones are raised by \(\gamma^{ab}\) and by \(\gamma_{ab}\), respectively. Let \(h = - \det e_{\mu a}\), then \(h = \sqrt{-g}\), where \(g = \det g_{\mu \nu}\). For any derivation \(\partial\), from (2.2) we have

\[\partial h^{a b} = - h^{a b} \partial e_{\mu a}, \quad (2.3)\]

\[\partial h = h^{a b} \partial e_{\mu a} \partial e_{\nu b}. \quad (2.4)\]

Expressing the affine connection \(\Gamma_{\mu \nu}^a\) in terms of vierbein, we see\(^*\)

\[\Gamma_{\mu \nu}^a M^a_{\lambda} = h^{a b} \partial e_{\mu a} \partial e_{\nu b} M^b_{\lambda}. \quad (2.5)\]

for any \(M^a = M^b\).

The generally-covariant Dirac \(\gamma\)-matrices are defined by

\[\gamma^a = h^{a b} \dot{\gamma}_b, \quad (2.6)\]

so that \(\{\gamma^a, \gamma^b\} = 2\eta^{a b}\), where \(\dot{\gamma}_b\)'s denote the usual \(\gamma\)-matrices in the flat space-time.

The spin affine connection is defined by

\[\Gamma_{\mu}^a = \frac{1}{2} \dot{\sigma}_{a b} \Gamma_{\mu}^{a b}, \quad (2.7)\]

\(^*\) A middle dot indicates that the preceding differential operator does not act beyond it.
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where \( \delta_{ab} = \left( \hat{\Gamma}_{ab} - \hat{\Gamma}_{ba} \right) / 4 \) and

\[
\Gamma^a_{\mu \nu} = (\partial_\mu h^a_{\nu} - \Gamma^a_{\nu \rho} h^\rho_{\mu} \big) h^{\mu a} = \frac{1}{2} \left\{ \left[ h^{b \rho} \partial_\mu h^a_{\nu} - h^{b \rho} \partial_\nu h^a_{\mu} - h^{a \rho} h^{b \eta} \partial_\mu h^{\eta}_{\nu} \big] \right\} - (a \leftrightarrow b) - \Gamma^a_{\nu \mu} \,. \quad (2.8)
\]

Then the covariant derivative of \( \gamma^a \) vanishes:

\[
\Gamma_{\mu} \gamma^a + \Gamma_{\mu} \gamma^a - [\Gamma_{\mu}, \gamma^a] = 0 \, . \quad (2.9)
\]

The Dirac field \( \psi \) satisfies \(^3\)

\[
(\gamma^a \Gamma_{\mu} + im) \psi = 0 \, , \quad (2.10)
\]

where \( m \) stands for a mass and \( \Gamma_{\rho} \psi = (\partial_\rho - \Gamma_\rho) \psi \). The conjugate field \( \bar{\psi} \) is defined by \( \bar{\psi} = \psi^\dagger \bar{\gamma}_\nu \). Since \( h^{ab} \) is hermitian, \( (2.6) \) implies \( \bar{\gamma}_\nu \gamma^a \bar{\gamma}_\nu = \gamma^a \). Hence \( (2.10) \) implies

\[
\bar{\psi} (\bar{\Gamma}_{\mu} \gamma^a - im) = 0 \quad (2.11)
\]

with \( \bar{\psi} \bar{\Gamma}_{\mu} = \bar{\psi} (\partial_\mu + \Gamma_\mu) \). The Dirac equations \( (2.10) \) and \( (2.11) \) follow from the following Dirac Lagrangian density \( \mathcal{L}_D \) as is seen by using \( (2.9) \):

\[
\mathcal{L}_D = \frac{1}{2} m \hbar \bar{\psi} (\gamma^a \partial_\mu - \partial_\mu \gamma^a - \{ \gamma^a, \Gamma_\mu \}) \psi = \frac{1}{2} m \hbar \bar{\psi} \psi \,. \quad (2.12) \)

It can be shown that \( \mathcal{L}_D \) is invariant under both the general coordinate transformation and the LL one.

In the Dirac theory, the canonical energy-momentum tensor density, \(^**\)

\[
T_{D \mu \nu}^{\gamma a} = [\partial \mathcal{L}_D / \partial (\partial_\epsilon \psi)] \partial_\mu \psi - \partial_\mu \bar{\psi} [\partial \mathcal{L}_D / \partial (\partial_\epsilon \bar{\psi})] = \partial_\mu \bar{\psi} \mathcal{L}_D
\]

\[
= \frac{1}{2} m \hbar \bar{\psi} (\gamma^a \bar{\gamma}_\nu - \gamma^a \bar{\gamma}_\nu) \psi - \partial_\mu \bar{\psi} \mathcal{L}_D \, , \quad (2.13)
\]

is different from the gravitational-source energy-momentum tensor density \( T_D^{\gamma a} \) defined by

\[
T_D^{\gamma a} = - m \hbar \bar{\psi} \left( \frac{\partial \mathcal{L}_D}{\partial h_{\mu \nu}} - \partial_\mu \bar{\psi} \left( \frac{\partial \mathcal{L}_D}{\partial (\gamma^a h_{\mu \nu})} \right) \right) = - g^{\gamma a} \mathcal{L}_D + \frac{1}{2} m \hbar \bar{\psi} (\gamma^a \bar{\gamma}_\nu - \gamma^a \bar{\gamma}_\nu) \psi
\]

\[+ \frac{1}{16} m \hbar \bar{\psi} (\{ \gamma^a, [\gamma^b, [\gamma^c, [\gamma^d, [\gamma^e, \gamma^f]]] \}) \bar{\gamma}_\nu + \bar{\gamma}_\nu (\{ \gamma^a, [\gamma^b, [\gamma^c, [\gamma^d, \gamma^e]]] \}) \psi \,. \quad (2.14)
\]

\(^{*}\) The differential operator \( \delta_\psi \) acts on \( \bar{\psi} \) but not on \( h \).

\(^{**}\) In the present paper, we always consider energy-momentum tensor densities instead of tensors.
This expression for $T_D^{\rho\sigma}$ looks non-symmetric in its appearance under $\mu \leftrightarrow \nu$, but with the aid of the Dirac equations, it can be rewritten into a symmetric form:

$$T_D^{\rho\sigma} = \frac{1}{4} i \hbar \bar{\psi} (\gamma^\rho \bar{\psi} + \gamma^\sigma \bar{\psi} - \bar{\psi} \gamma^\rho - \bar{\psi} \gamma^\sigma) \psi. \quad (2.15)$$

This fact is a consequence of the LL invariance of $\mathcal{L}_D$.

Finally, we review some important consequences of the invariant variation theory (the second Noether theorem) for later convenience.

Let $A$ be any scalar density depending on some fields $\Phi_\alpha$. Then the invariance of $\int d^4x A$ under the general coordinate transformation implies that the following three identities hold:

$$\sum_\alpha \left\{ \partial_\rho [\partial_\sigma (A) [\Phi_\alpha]^{\mu}] + \partial_\sigma (A) \partial_\mu \Phi_\alpha \right\} = 0, \quad (2.16)$$

$$\sum_\alpha \left[ \partial A / \partial (\partial_\rho \Phi_\alpha) \right] \partial_\nu \Phi_\alpha - \partial_\nu A = \sum_\alpha \partial_\sigma (A) [\Phi_\alpha]^{\mu} - \partial_\mu K^{\rho\nu} = 0, \quad (2.17)$$

$$K^{\rho\nu} = - K^{\nu\rho} \quad (2.18)$$

with

$$K^{\rho\nu} = - \sum_\alpha \left[ \partial A / \partial (\partial_\rho \Phi_\alpha) \right] [\Phi_\alpha]^{\nu}, \quad (2.19)$$

where $\partial_\sigma \Phi_\alpha$ denotes the Euler derivative with respect to $\Phi_\alpha$ and $[\Phi_\alpha]^{\nu}$ is the transformation matrix of $\Phi_\alpha$ under the infinitesimal general coordinate transformation (i.e., the infinitesimal change of $\Phi_\alpha$ is written as $[\Phi_\alpha]^{\nu} \partial_\nu \varepsilon$).

Since

$$[h_{\rho\mu}]^{\nu} = - \partial_\rho h_{\mu\nu}, \quad (2.20)$$

$$[\bar{\psi}]^{\nu} = [\bar{\phi}]^{\nu} = 0, \quad (2.21)$$

for $A = \mathcal{L}_D$ (2.16) becomes

$$\partial_\rho \left[ \partial_{h_{\rho\mu}} (\mathcal{L}_D) (- \partial_\sigma h_{\mu\nu}) \right] + \partial_{h_{\rho\mu}} (\mathcal{L}_D) \partial_\sigma h_{\mu\nu}$$

$$+ \partial_\rho \left( \mathcal{L}_D \right) \partial_\sigma \phi - \partial_\nu \bar{\phi} \cdot \partial_\sigma \left( \mathcal{L}_D \right) = 0. \quad (2.22)$$

The last two terms vanish because of the field equations for $\psi$ and $\bar{\phi}$, namely, (2.10) and (2.11). Therefore (2.22) reduces to

$$\partial_\rho T_D^{\rho\mu} - h^{\nu\mu} \partial_\nu h_{\rho\sigma} \cdot T_D^{\rho\sigma} = 0, \quad (2.23)$$

that is,

$$\Gamma_\rho T_D^{\rho\nu} = 0 \quad (2.24)$$

because of $T_D^{\rho\nu} = T_D^{\nu\rho}$ and (2.5).

The remaining identities (2.17) and (2.18) become
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\[ T^{\alpha \sigma}_{D} \rho_{i} + \partial_{\rho} h_{\nu a} \frac{\partial L}{\partial (\partial_{\rho} h_{\nu a})} - T_{D}^{\alpha \sigma}_{i} = \tilde{h}_{i a} \frac{\partial L}{\partial (\partial_{\rho} h_{\nu a})}, \]

\[ h_{i a} \frac{\partial L}{\partial (\partial_{\rho} h_{\nu a})} = - h_{i a} \frac{\partial L}{\partial (\partial_{\rho} h_{\nu a})}, \]

respectively. They are important in § 4.

§ 3. Lagrangian and field equations

We first introduce the LL-BRS transformation, which is denoted by \( \delta_{LL} \), while we denote by \( \delta \) the BRS transformation corresponding to the general coordinate transformation as before.\(^3\) Since the LL transformations form a Lorentz group, we have\(^6\)

\[ \delta_{LL} (h_{\rho a}) = - t_{a}^{b} h_{\rho b}, \]

\[ \delta_{LL} (\phi) = - \frac{1}{2} t^{ab} \bar{\phi} \phi_{ab}, \]

\[ \delta_{LL} (t_{ab}) = - t_{a}^{c} t_{cb}. \]

where \( t_{ab} \) is one of the LL-FP ghosts. We also introduce another LL-FP ghost \( t_{ab} \) and an auxiliary boson field \( s_{ab} \). As usual,\(^3,11\) we assume that

\[ \delta_{LL} (t_{ab}) = i s_{ab}, \]

\[ \delta_{LL} (s_{ab}) = 0. \]

The three fields \( s_{ab}, t_{a} \) and \( t_{ab} \) are all hermitian and antisymmetric under \( a \leftrightarrow b \). Furthermore, they are assumed to be BRS-invariant \( (\delta (s_{ab}) = 0, \text{ etc.}) \). The LL-FP ghosts \( t_{ab} \) and \( t_{ab} \) are fermion fields. They may be commutative or anticommutative with the FP ghost \( \bar{c} \) and \( c \) and with the Dirac fields \( \psi \) and \( \bar{\psi} \). But, for definiteness, we assume that all fermion fields are mutually anticommutative (at the classical level). Hence \( \{\delta, \delta_{LL}\} = 0 \). In contrast with the BRS transformation \( \delta \), the LL-BRS transformation \( \delta_{LL} \) commutes with \( \partial_{\rho} \) because the LL invariance is an internal symmetry.

All the “old” fields \( g_{\mu}, b_{\mu}, c^{a} \) and \( \bar{c}_{\mu} \), which have been considered previously,\(^3,4\) are LL-BRS-invariant. Indeed, we can immediately confirm that

\[ \delta_{LL} (g_{\mu}) = 0 \]

from (3.1) and (2.1). Hence \( \delta_{LL} (h) = 0 \). We can also easily show that

\[ \delta_{LL} (\Gamma_{\mu}^{ab}) = - t_{a}^{c} \Gamma_{\mu c}^{b} + t_{b}^{c} \Gamma_{\mu c}^{a} - \partial_{\mu} t_{ab}, \]

\[ \delta_{LL} (L_{D}) = 0. \]

\(^3\) \( \delta (h_{\rho a}) = \kappa \tilde{\partial}_{\rho} \epsilon_{i} \cdot h_{i a}, \delta (\phi) = \delta (\bar{\psi}) = 0.\)
Thus both the old Lagrangian density,*) \( \mathcal{L}_D^\prime \), considered previously and the Dirac one \( \mathcal{L}_D \) are LL-BRS-invariant.

Now, we introduce the gauge-fixing term for the LL transformation. It must meet the following requirements:

1. It should be non-invariant under the LL-BRS transformation.
2. It should depend only on \( s_{ab} \) and \( h_{\alpha\beta} \).
3. When multiplied by \( h^{-1} \), it should be BRS-invariant.
4. The number of \( \partial_\alpha \) involved in any term of it should not exceed two.
5. It should involve the six degrees of freedom of \( \hat{h}_{\alpha\beta} \) independent of \( \hat{g}_{\alpha\beta} \).

Then the following choice of the LL-gauge-fixing Lagrangian density is the simplest possible and most natural one:

\[
\mathcal{L}_{\text{LLGF}} = h g^{\mu\nu} F^a_{\nu} \partial_\mu s_{ab} \quad (3.9)
\]

The corresponding LL-FP-ghost term \( \mathcal{L}_{\text{LLFP}} \) is determined by

\[
\mathcal{L}_{\text{LL}} = \mathcal{L}_{\text{LLGF}} + \mathcal{L}_{\text{LLFP}} = -i \delta_{\text{LL}}(h g^{\mu\nu} F^a_{\nu} \partial_\mu s_{ab}) \quad (3.10)
\]

that is,

\[
\mathcal{L}_{\text{LLFP}} = -i h g^{\mu\nu} \partial_\mu s_{ab} (e^{a}_{\epsilon} F^\epsilon_{\nu} - e^{b}_{\epsilon} F^\epsilon_{\nu} + \partial_\epsilon e^{ab}) \quad (3.11)
\]

The total Lagrangian density,

\[
\mathcal{L}_{\text{tot}} = \mathcal{L} + \mathcal{L}_D + \mathcal{L}_{\text{LL}} \quad (3.12)
\]

is invariant under the LL-BRS transformation, and the total action is BRS-invariant.

The field equations which follow from (3.12) are as follows. The Einstein equation* becomes

\[
h (R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R - B^{\mu\nu}) = -\kappa (T_D^{\mu\nu} + T_{\text{LL}}^{\mu\nu}) \quad (3.13)
\]

where \( R^{\mu\nu} \) is the Ricci tensor \( (R = R^\mu_\mu) \) and

\[
B^{\mu\nu} = (g^{\mu\omega} g^{\nu\rho} - \frac{1}{2} g^{\mu\nu} g^{\rho\omega}) \left[ (\partial_\rho h_\lambda - i \kappa \partial_\nu \bar{c}_\rho \cdot \partial_\lambda c^\rho) + (\sigma \leftrightarrow \sigma) \right] \quad (3.14)
\]

\[
T_{\text{LL}}^{\mu\nu} = -h_a^\mu \left( \frac{\partial \mathcal{L}_{\text{LL}}}{\partial h^\mu_{\rho\alpha}} - \partial_\beta \frac{\partial \mathcal{L}_{\text{LL}}}{\partial (\partial_\beta h^\mu_{\rho\alpha})} \right) \quad (3.15)
\]

The following field equations* remain unchanged:

\[
\partial_\mu (h g^{\mu\nu}) = 0 \quad (3.16)
\]

\[
g^{\mu\nu} \partial_\mu \bar{c} = 0 \quad (3.17)
\]

\[
g^{\mu\nu} \partial_\nu \bar{c} = 0 \quad (3.18)
\]

*) We consider the Landau-gauge case alone and, for simplicity, omit any matter field other than the Dirac field.
New field equations are
\[ g^{\alpha \beta} \partial_{\mu} \Gamma_{\nu}^{\alpha \beta} = 0, \]  
\[ g^{\alpha \beta} \left( \partial_{\mu} t_{\alpha}^{c} \cdot \Gamma_{\nu}^{\alpha \beta} + \partial_{\mu} t_{\beta}^{a} \cdot \Gamma_{\nu}^{\alpha \beta} \right) = 0, \]  
\[ g^{\alpha \beta} \left( \partial_{\mu} j_{\alpha}^{a} \cdot \Gamma_{\nu}^{\alpha \beta} + \partial_{\mu} j_{\beta}^{a} \cdot \Gamma_{\nu}^{\alpha \beta} \right) = 0. \]

Together with the Dirac equations (2.10) and (2.11).

From (3.7), we see that (3.20) coincides with the \( \delta_{LL} \) of (3.19). Likewise, the \( \delta_{LL} \) of (3.21) is
\[ ig^{\alpha \beta} \left( \partial_{\mu} s_{\alpha}^{c} \cdot \Gamma_{\nu}^{\alpha \beta} + \partial_{\mu} s_{\beta}^{a} \cdot \Gamma_{\nu}^{\alpha \beta} + \partial_{\mu} s_{\alpha}^{a} \cdot \Gamma_{\nu}^{\alpha \beta} \right) + \left[ g^{\alpha \beta} \partial_{\mu} s_{\alpha}^{c} \cdot \left( t_{\alpha}^{d} \Gamma_{\nu}^{\alpha \beta} \right) - t_{\beta}^{b} \Gamma_{\nu}^{\alpha \beta} + \partial_{\mu} s_{\alpha}^{a} \right] \]  
\[ = 0. \]

It is quite instructive to see that (3.22) coincides with the antisymmetric part of the field equation (3.13).

Since \( T^{\alpha \mu \nu \beta}_{D} \) is symmetric, the antisymmetric part of (3.13) is
\[ T_{\alpha \mu \nu \beta} = T_{LLG_{\alpha \mu \nu \beta}^{\beta}} + T_{L_{\alpha \mu \nu \beta}^{\beta}}. \]

Here
\[ T_{LLG_{\alpha \mu \nu \beta}^{\beta}} = T_{LLFF_{\alpha \mu \nu \beta}^{\beta}} \]

with
\[ h_{\alpha}^{b} h_{\beta}^{a} T_{LLG_{\alpha \mu \nu \beta}^{\beta}} = - h_{\mu}^{b} \left( \frac{\partial L_{LLG}}{\partial h_{\mu a}} - \frac{\partial L_{LLG}}{\partial (\partial_{\mu} h_{\alpha}^{a})} \right) \]
\[ = - g^{ab} L_{LLG} + h (h_{\alpha}^{a} h_{\beta}^{b} + h_{\beta}^{a} h_{\alpha}^{b}) \Gamma_{\nu}^{cd} \partial_{\alpha} s_{\beta}^{d} \]
\[ - h g^{ab} h_{\beta}^{a} \frac{\partial \Gamma_{\nu}^{cd}}{\partial s_{\alpha}^{d}} + h_{\mu}^{b} \frac{\partial}{\partial (\partial_{\mu} h_{\alpha}^{a})} \frac{\partial s_{\alpha}^{d}}{\partial h_{\beta}^{a}} \]
\[ = \frac{\partial L_{LLFF}}{\partial h_{\mu a}} - \frac{\partial L_{LLFF}}{\partial (\partial_{\mu} h_{\alpha}^{a})} \]
\[ = - g^{ab} L_{LLFF} - i h (h_{\alpha}^{a} h_{\beta}^{b} + h_{\beta}^{a} h_{\alpha}^{b}) \partial_{\alpha} s_{\beta}^{d} \Gamma_{\nu}^{cd} \]
\[ + 2 i h g^{ab} \partial_{\alpha} s_{\beta}^{d} \Gamma_{\nu}^{cd} \]
\[ = - 2 i h_{\mu}^{a} \partial_{\alpha} \left[ h_{\alpha}^{b} \partial_{\beta} h_{\mu}^{a} \right] \frac{\partial \Gamma_{\nu}^{cd}}{\partial (\partial_{\mu} h_{\alpha}^{a})}. \]

The explicit expressions
\[ 2 \partial \Gamma_{\alpha \mu \nu \beta}^{cd} / \partial h_{\alpha}^{a} = \left[ - h_{\mu}^{a} h_{\alpha}^{b} \partial_{\beta} h_{\nu}^{e} + h_{\mu}^{a} h_{\alpha}^{b} \partial_{\beta} h_{\nu}^{e} \right. \]
\[ + h_{\mu}^{a} h_{\alpha}^{b} h_{\nu}^{c} \partial_{\beta} h_{\nu}^{e} + h_{\mu}^{a} h_{\alpha}^{b} h_{\nu}^{c} \partial_{\beta} h_{\nu}^{e} \]
\[ - \partial_{\alpha} h_{\mu}^{a} h_{\alpha}^{b} \partial_{\beta} h_{\nu}^{e} \right] \]  
\[ = (c \leftrightarrow d), \]
yield the following useful identities:

\[
\begin{align*}
2\partial \Gamma^a_{\nu\rho} / \partial (\partial_\rho \eta^a) &= [\dot{\partial}_\nu \eta^a \eta^a - \dot{\partial}_\nu \eta^a \eta^a] - (c \leftrightarrow d) \\
\Gamma^a_{\nu\rho} &= \frac{\partial \Gamma^a_{\nu\rho}}{\partial (\partial_\rho \eta^a)} (a \leftrightarrow b) \quad (3.29)
\end{align*}
\]

With the aid of them together with (3.16), we can show that

\[
\begin{align*}
\Gamma^a_{\nu\rho} &= 2h^a \eta^a \Gamma^a_{\nu\rho} - (a \leftrightarrow b) \\
&= 2h^a \eta^a (\partial_\epsilon \eta^a \cdot \Gamma^a_{\nu\rho} - \partial_\eta \eta^a \cdot \Gamma^a_{\nu\rho} + \partial_\epsilon \eta^b \cdot \eta^a \cdot \epsilon^b) \quad (3.31)
\end{align*}
\]

Thus (3.22) is equivalent to (3.23).

Finally, we show that the covariant derivative of \( T_{LL}^\alpha \) vanishes. Applying the same reasoning as the one at the end of § 2 to \( \mathcal{A} = \mathcal{L}_{LL} \), we see that (2.16) yields

\[
\begin{align*}
\partial_\alpha T_{LL}^\alpha &= h^a \partial_\alpha \eta^a \cdot T_{LL}^\alpha = 0. \\
\text{(3.33)}
\end{align*}
\]

Because of (3.23), the antisymmetric part of \( T_{LL}^\alpha \) vanishes. Hence (3.33) implies

\[
\begin{align*}
\mathcal{F}_\alpha T_{LL}^\alpha &= 0. \\
\text{(3.34)}
\end{align*}
\]

Therefore, the covariant derivative of (3.13) yields

\[
\begin{align*}
g^a \partial_\alpha \eta^a \cdot b_\alpha &= 0. \\
\text{(3.35)}
\end{align*}
\]

We note that (3.35) is a direct consequence of the BRS invariance, because it is essentially the \( \mathcal{F} \) of (3.18).

§ 4. Conserved quantities

In our theory, there are many conserved currents in addition to the Poincaré generators. First, the Dirac current \( J_\nu^a \) is given by

\[
J_\nu^a = h^a \eta^a \psi, \\
\text{(4.1)}
\]

which plays no essential role in our formalism.

Next, the LL-FP-ghost current \( J^a \) is defined by
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\[ J_i^\mu = [\partial \mathcal{L}_{LL}/\partial (\partial_{\mu} t^{ab})]_{t^{ab}} + i g^{\mu \nu} (\partial_{\nu} t^{ab} - \partial_{\nu} t^{ab} - t^{ab} + 2 i g^{\mu \nu} \Gamma_{\nu}^{\alpha} t_{ab} \epsilon_{\alpha} c) \]

(4.2)

It is easy to confirm \( \partial_{\mu} J_i^\mu = 0 \) by means of (3.20) and (3.21).

The LL-BRS current \( J_i^\mu \) is defined by

\[ J_i^\mu = \sum_{A} \delta_{LL}(\Phi_A) \left[ \partial \left( \partial_{\mu} \Phi_A \right) \right] \partial \mathcal{L}_{tot}, \]

(4.3)

where the summation goes over all fields. If \( A \) is a quantity whose dependence on \( h_{\mu \nu} \) is only through \( g_{\mu \nu} \), then we see

\[ \delta_{LL}(h_{\mu \nu}) \frac{\partial A}{\partial (\partial_{\mu} h_{\alpha \beta})} = - g^{\mu \nu} \left( \frac{\partial A}{\partial (\partial_{\mu} g_{\alpha \beta})} + \frac{\partial A}{\partial (\partial_{\mu} g_{\alpha \beta})} \right) = 0 \]

(4.4)

owing to the antisymmetry of \( t_{ab} \). Hence \( J_i^\mu \) receives no contribution from \( \mathcal{L} \).

After some calculation, we find

\[ J_i^\mu = i g^{\mu \nu} (s_{AB} \partial_{\nu} t_{ab} - \partial_{\nu} s_{AB} \cdot t_{ab} + 2 i g^{\mu \nu} \epsilon_{\alpha} c) \]

\[ = - \delta_{LL}(J_i^\mu). \]

(4.5)

Of course, one can directly confirm \( \partial_{\mu} J_i^\mu = 0 \) by using (3.20) \( \sim (3.22) \).

Since \( \mathcal{L}_{D} + \mathcal{L}_{LL} \) is independent of the FP ghosts \( e^\mu \) and \( \epsilon^\mu \), the FP-ghost current \( J_i^\mu \) remains unchanged.

One naturally expects also that the BRS current \( J_i^\mu \) remains unchanged because we still have (3.17) and (3.35). But the validity of this statement is non-trivial, because the BRS Noether current receives non-vanishing contributions from \( \mathcal{L}_{D} \) and \( \mathcal{L}_{LL} \). Indeed, the additional contribution [see (4.1) of Ref. 2] is

\[ - \kappa (\partial c^i h_{\alpha \beta} + c^i \partial h_{\alpha \beta}) \partial (\mathcal{L}_{D} + \mathcal{L}_{LL}) / \partial (\partial_{\mu} h_{\alpha \beta}) - \kappa c^i (T_{D,}^{a, i} + T_{a, i}^{a,}) \]

(4.6)

where \( T_{D,}^{a, i} \) is defined by (2.13) and

\[ T_{a, i}^{a,} = [\partial \mathcal{L}_{LL}/\partial (\partial_{\mu} t^{ab})]_{t^{ab}} + [\partial \mathcal{L}_{LL}/\partial (\partial_{\mu} t^{ab})]_{t^{ab}} \]

(4.7)

\[ + [\partial \mathcal{L}_{LL}/\partial (\partial_{\mu} t^{ab})]_{t^{ab}} - \delta_{\mu} \mathcal{L}_{LL}. \]

In order to simplify the expression for the BRS Noether current, we made use of the Einstein equation.\(^{p}\) The Einstein equation is now (3.13), which contains additional terms \( \kappa (T_{D}^{a, i} + T_{a, i}^{a,}) \). Therefore the total change of the expression for the BRS Noether current is the sum of (4.6) and

\[ \kappa c^i (T_{D,}^{a, i} + T_{a, i}^{a,}) \]

(4.8)

Because of its dependence on \( c^i \), it can be written as a total divergence of an antisymmetric tensor density if and only if
As explained at the end of § 2, they are indeed identities. More precisely, (4·9) and (4·10) hold, without using any field equation, for each contribution from \( \mathcal{L}_D, \mathcal{L}_{LLGF} \) and \( \mathcal{L}_{LLFP} \) separately. Their validity can also be directly confirmed by means of the formulae

\[
\frac{\partial \Gamma^{\sigma \rho}}{\partial (\vartheta_{\mu} h_{\nu a})} = - \frac{\partial \Gamma^{\rho \sigma}}{\partial (\vartheta_{\nu} h_{\mu a})},
\]

\[
h_{\nu a} \frac{\partial \Gamma^{\sigma \rho}}{\partial h_{\mu a}} + (\partial_{\mu} h_{\nu a} - \partial_{\nu} h_{\mu a}) \frac{\partial \Gamma^{\rho \sigma}}{\partial (\vartheta_{\nu} h_{\mu a})} = \delta^{\sigma \rho} \delta^{\nu \mu}.
\]

Thus the expression for the BRS current\(^7\) \( J_{\mu}^a \) remains unchanged.

Quite a similar mechanism takes place also for the (total) canonical energy-momentum tensor density\(^8\)

\[
\mathcal{T}^a_i = \sum A [\vartheta \partial \mathcal{L}_{\text{tot}} / \partial (\vartheta A)] \partial \vartheta A - \delta^a_i \mathcal{L}_{\text{tot}}.
\]

Its effective additional contribution is

\[
\frac{\partial (\mathcal{L}_D + \mathcal{L}_{LL})}{\partial (\vartheta_{\mu} h_{\nu a})} \vartheta_{\mu} h_{\nu a} + (T_{\text{can}}^{\mu a} + T_{\text{can}}^{\nu a}) - (T_{\text{can}}^{\mu a} + T_{\text{can}}^{\nu a}),
\]

which is precisely equal to \( \partial_{\nu} K_{\nu a} \), as is shown above. Thus the expression for the translation generator,\(^9\)

\[
P_\mu = k^{-1} \int d^4 x \ h^\mu \partial_{\nu} h_{\nu} \vartheta_{\mu} \vartheta_{\mu},
\]

remains unchanged.

Finally, the canonical angular-momentum tensor density is defined by\(^9\)

\[
\mathcal{M}^a_{\mu
u} = x^a \mathcal{T}^I - x^\mu \mathcal{T}^I_{\nu} + \mathcal{S}^I_{\mu
u},
\]

where \( \mathcal{S}^I_{\mu
u} \) stands for the spin angular momentum. Under the true Lorentz transformation, \( \vartheta \) should transform like a spinor. Correspondingly, the vierbein \( h_{\mu a} \) should transform not like a vector but like a tensor. In general, any Latin index should not be distinguished from a Greek one under the Lorentz transformation.
Hence we must be very careful about raising and lowering indices.

Since
\[
\left( \eta_{\mu \rho} h_{\nu \alpha} - \eta_{\nu \alpha} h_{\mu \rho} + \eta_{\rho \mu} h_{\nu \alpha} - \eta_{\nu \alpha} h_{\rho \mu} \right) \partial \phi / \partial (\partial_{\mu} h_{\nu \alpha})
\]
\[
= \left( \eta_{\mu \rho} g_{\nu \alpha} - \eta_{\nu \alpha} g_{\mu \rho} \right) \left[ \partial \phi / \partial (\partial_{\nu} g_{\alpha \mu}) + \partial \phi / \partial (\partial_{\mu} g_{\alpha \nu}) \right]
\]
(4.18)
owing to the cancellation of the last two terms, the additional contribution to \( S_{\mu \nu} \) is
\[
\left( \eta_{\mu \rho} h_{\nu \alpha} - \eta_{\nu \alpha} h_{\mu \rho} + \eta_{\rho \mu} h_{\nu \alpha} - \eta_{\nu \alpha} h_{\rho \mu} \right) \partial (\mathcal{L}_D + \mathcal{L}_{LL}) / \partial (\partial_{\mu} h_{\nu \alpha})
\]
\[
+ \left[ \partial \mathcal{L}_D / \partial (\partial_{\nu} \phi) \right] \delta_{\rho \mu} \phi + \delta_{\rho \mu} \left[ \partial \mathcal{L}_D / \partial (\partial_{\nu} \phi) \right] + S_{\mu \nu}^{LL}
\]
(4.19)
with
\[
S_{\mu \nu}^{LL} := \left( \eta_{\mu \nu} \phi_{\nu \alpha} + \eta_{\nu \alpha} \phi_{\mu \nu} - \eta_{\alpha \nu} \phi_{\mu \nu} \right) \partial \mathcal{L}_{LL} / \partial (\partial_{\mu} h_{\nu \alpha})
\]
\[- \left( \eta_{\mu \nu} \phi_{\nu \alpha} + \eta_{\nu \alpha} \phi_{\mu \nu} - \eta_{\alpha \nu} \phi_{\mu \nu} \right) \partial \mathcal{L}_{LL} / \partial (\partial_{\rho} h_{\nu \alpha})
\]
\[- \left( - \eta_{\mu \nu} \phi_{\nu \alpha} + \eta_{\nu \alpha} \phi_{\mu \nu} - \eta_{\alpha \nu} \phi_{\mu \nu} \right) \partial \mathcal{L}_{LL} / \partial (\partial_{\rho} h_{\nu \alpha})
\]
(4.20)
On the other hand, the orbital angular momentum \( x_\mu \delta_{\mu \nu} K^{\nu \rho} \) acquires an additional contribution
\[
x_\rho \partial_{\nu} K^{\nu \rho} - x_\nu \partial_{\rho} K^{\nu \mu},
\]
(4.21)
That is, apart from a total divergence of a quantity antisymmetric under \( \mu \leftrightarrow \nu \), \( M_{\mu \nu} \) receives the contribution
\[
- \eta_{\mu \rho} K^{\rho \nu} + \eta_{\nu \rho} K^{\rho \mu} = \left( - \eta_{\rho \mu} h_{\nu \alpha} + \eta_{\nu \alpha} h_{\rho \mu} \right) \partial \mathcal{L}_D / \partial (\partial_{\mu} h_{\nu \alpha})
\]
(4.22)
which exactly cancels the first two terms of (4.19). Furthermore, by direct calculation, we find
\[
\left( \eta_{\mu \rho} h_{\nu \alpha} - \eta_{\nu \alpha} h_{\mu \rho} \right) \partial \mathcal{L}_D / \partial (\partial_{\mu} h_{\nu \alpha})
\]
\[
= - \frac{1}{2} i h \tilde{\phi} \{ j^\mu, \delta_{\mu \nu} \} \phi
\]
\[
= - \left[ \partial \mathcal{L}_D / \partial (\partial_{\mu} \phi) \right] \delta_{\mu \nu} \phi - \tilde{\phi} \delta_{\mu \nu} \left[ \partial \mathcal{L}_D / \partial (\partial_{\nu} \phi) \right].
\]
(4.23)
(This identity is a consequence of the LL invariance of \( \mathcal{L}_D \).) Thus no contribution from \( \mathcal{L}_D \) remains.\(^{*}\)

After all, the Lorentz generator\(^{\dagger}\) \( M_{\mu \nu} \) acquires an extra contribution
\[
M_{\mu \nu} = \int d^4 x \left[ \left( \eta_{\mu \nu} h_{\rho \sigma} - \eta_{\rho \sigma} h_{\mu \nu} \right) \partial \mathcal{L}_{LL} / \partial h_{\rho \sigma} + S_{\mu \nu} \right]
\]
\[
= 2 \int d^4 x \ h g^{\rho \sigma} L_{\rho \sigma},
\]
(4.24)
\(^{*}\) In the flat space-time, this corresponds to the well-known fact that the angular momentum tensor contains no extra spin term when expressed in terms of the symmetric energy-momentum tensor.

\(^{\dagger}\) In the flat space-time, this corresponds to the well-known fact that the angular momentum tensor contains no extra spin term when expressed in terms of the symmetric energy-momentum tensor.
where
\[
L_{a}^{ab} \equiv \partial_{a}s^{ab} - \Gamma_{a}^{\cdots e} s_{b}^{\cdots e} + \Gamma_{a}^{\cdots e} s_{e}^{\cdots b} - i\left[\Gamma_{a}^{\cdots e} (\partial_{e} s_{b}^{ab} - \Gamma_{e}^{\cdots d} s_{d}^{ab} - \Gamma_{e}^{\cdots d} s_{d}^{ab}) - (a \leftrightarrow b)\right].
\]

Since
\[
\partial_{a} (\hbar g^{\alpha} L_{s_{a}}^{ab}) = 0
\]
owing to (3.16), (3.19), (3.20) and (3.22), we see that $\mathcal{M}_{L_{s}}$ is a conserved quantity.

§ 5. Asymptotic fields

Since it is inadequate to eliminate $s_{ab}$ from our Lagrangian by integrating by parts, the six components of $s_{ab}$ must be regarded as canonical variables, that is, $s_{ab}$ is not a Lagrange multiplier field. For $J_{D}$, we should eliminate $\partial_{a} b_{b}$ by integrating by parts. Thus the canonical variables are $h_{a}, c_{v}, \bar{c}_{v}, \bar{a}_{a}, l_{a}, l_{b}$ and $\psi$. Canonical quantization can be carried out consistently. Detailed analysis will be presented in a succeeding paper. We shall show there that all commutation relations concerning the old fields $(g_{\mu
u} = h_{a}, b_{a}, c_{v}, \bar{c}_{v})$ are precisely reproduced.

The physical states are defined by the subsidiary conditions
\[
Q_{b}|\text{phys}\rangle = 0, \quad Q_{a}|\text{phys}\rangle = 0,
\]
where, of course, both $Q_{b} = \int d^{3}x J_{b}^{a}$ and $Q_{a} = \int d^{3}x J_{a}^{a}$ are conserved.

In order to show the positive semi-definiteness of the norm of the physical-state subspace, we investigate the asymptotic fields under the postulate of asymptotic completeness.

We introduce the asymptotic fields by
\[
(h_{a} - \eta_{a}) / 2\sqrt{\kappa} \rightarrow \chi_{a} + \text{Källén term},
\]
\[
b_{a} / \sqrt{\kappa} \rightarrow \vec{b}_{a}, \quad c_{v} \rightarrow \vec{c}_{v}, \quad \bar{c}_{v} \rightarrow \vec{\bar{c}}_{v},
\]
\[
\sqrt{\kappa} a_{a b} \rightarrow \bar{a}_{a b}, \quad l_{a b} \rightarrow \tau_{a b}, \quad \bar{l}_{a b} \rightarrow \bar{\tau}_{a b},
\]
\[
\psi \rightarrow \psi_{\text{asym}}
\]
as $x^{a} \rightarrow -\infty$ (or $x^{a} \rightarrow +\infty$). It is convenient to set
\[
\varphi_{a b} = \chi_{a b} + \chi_{b a} = \varphi_{b a},
\]
\[
\epsilon_{a b} = \chi_{a b} - \chi_{b a} = -\epsilon_{b a},
\]
so that $\varphi_{\mu
u}$ coincides with the asymptotic field of $g_{\mu\nu}$.

As before,\(^{1+6}\) we assume that the properties of the asymptotic fields are gov-

\^{1+6}\) Here we omit $Z$-factors and neglect the problem of ultraviolet divergence.
Earned by the linearized Lagrangian density of $\mathcal{L}_{\text{tot}}$ except for the renormalization of the parameters involved. The asymptotic Lagrangian density, $\mathcal{L}_{\text{asym}}$, corresponding to $\mathcal{L}$ remains unchanged. The total asymptotic-field Lagrangian density is given by

$$
\mathcal{L}_{\text{tot}}^{\text{asym}} = \mathcal{L}_{\text{asym}} = \bar{G}_{\text{asym}} - 2\partial^c \varphi \cdot \partial_c \sigma_{ab} - 2\partial^c \varphi \cdot \partial_c \sigma_{ab} - i\partial e_{ab} \cdot \partial^c \varphi_{a}^{bc} + \bar{\varphi}_{\text{asym}} \left[ \frac{1}{2} i \gamma^a (\bar{\sigma}_a - \bar{\sigma}_a) - m^2 \right] \varphi_{\text{asym}},
$$

(5.5)

from which we have

$$
\square \epsilon_{ab} = 0,
$$

(5.6)

$$
\square \sigma_{ab} = 0,
$$

(5.7)

By using the linearized De Donder condition

$$
\bar{\sigma}_c \varphi^{ac} = \frac{1}{2} \partial^c \varphi_c^{ac},
$$

(5.8)

(5.6) is simplified into

$$
\square \epsilon_{ab} = 0.
$$

(5.9)

Owing to the third term of (5.5), the field equation for $\varphi_{ab}$ is modified into

$$
\square \varphi_{ab} = \partial_a \partial_b \varphi + \partial_a \partial_b \sigma_{ac} + \partial_a \partial_b \sigma_{bc}.
$$

(5.10)

It is straightforward to analyze the canonical commutation relations for the asymptotic fields. We find that the four-dimensional commutation relations between the old fields remain unchanged. Those which involve the new fields are found to be

$$
[\varphi_{ab}(x), \epsilon_{cd}(y)] = [\varphi_{ab}(x), \sigma_{cd}(y)] = 0,
$$

(5.11)

$$
[\epsilon_{ab}(x), \sigma_{cd}(y)] = -\frac{1}{2} i (\gamma_{ac} \gamma_{bd} - \gamma_{ad} \gamma_{bc}) D(x - y),
$$

(5.12)

$$
[\epsilon_{ab}(x), \epsilon_{cd}(y)] = [\sigma_{ab}(x), \sigma_{cd}(y)] = 0,
$$

(5.13)

$$
\{\tau_{ab}(x), \bar{\tau}_{cd}(y)\} = \{\tau_{ab}(x), \bar{\tau}_{cd}(y)\} = 0,
$$

(5.14)

$$
\{\tau_{ab}(x), \tau_{cd}(y)\} = \{\tau_{ab}(x), \tau_{cd}(y)\} = 0,
$$

(5.15)

etc.

The expressions for $Q_s$ and $Q_t$ in terms of the asymptotic fields are, up to a multiplicative constant,

$$
Q_s = \int d^3 x \sigma_{ab} (\bar{\sigma}_a - \bar{\sigma}_a) \tau_{ab},
$$

(5.16)

$$
Q_t = i \int d^3 x \bar{\tau}_{ab} (\bar{\sigma}_a - \bar{\sigma}_a) \tau_{ab}.
$$

(5.17)

Hence
\[ [\varepsilon_{ab}, Q_a] = i\sigma_{ab}, \quad [\sigma_{ab}, Q_a] = 0, \quad (5.18) \]
\[ \{\varepsilon_{ab}, Q_a\} = 0, \quad \{\sigma_{ab}, Q_a\} = \sigma_{ab}. \quad (5.19) \]

Then applying the Kugo-Ojima theorem\(^9\), we see that the physical-state subspace is positive semidefinite.

Thus the physical S-matrix is unitary.

§ 6. Discussion

In the present paper, we have established that the quantum field theory of the coupled Einstein-Dirac system can be consistently formulated in the framework of the manifestly-covariant canonical formalism.

We make some remarks on the choice of the LL-gauge-fixing term. In the path-integral formalism, one can introduce almost any kind of the gauge-fixing term, though then gauge theories always suffer from the difficulty caused by the Gribov ambiguity.\(^9\) On the contrary, in the covariant canonical formalism, to which the Gribov ambiguity is totally irrelevant, the choice of the gauge-fixing term is quite restrictive. In our theory, our choice (3.9) is practically unique under the conditions stated in § 3. Simpler-looking choices,
\[
L'_{LLGF} = hh^{ab}\partial_i h^{bc}\partial_i s_{ab} \quad (6.1)
\]
and
\[
L''_{LLGF} = \partial_i (hh^{ab}) h^{bc}\partial_i s_{ab}, \quad (6.2)
\]
which are mutually equivalent, satisfy the first four conditions but not the last one. With (6.1) or (6.2), all canonical conjugates of \( h_{sa} \) are not independent. Of course, (3.9) is not unique in the mathematical sense, for instance, we may add \( L'_{LLGF} \) and/or \( h_{sa} s_{ab} \) to \( L_{LLGF} \). But such modifications are not interesting. We shall see in a succeeding paper that the LL-gauge-fixing term (3.9) yields quite natural equal-time commutation relations between Heisenberg fields.

References

5) See, e.g.,
7) See, e.g.,