Dynamics of the Noise-Induced Phase Transition of the Verhulst Model

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Langevin-type equation with a nonlinear drift term and a multiplicative noise term is treated. Dynamical aspects of noise-induced phase transition are studied with the aid of the method of asymptotic iteration (MAI) introduced in a previous paper. The temporal behavior of the first moment is solved exactly. It is shown that the critical slowing down does not occur at the so-called transition point.

§ 1. Introduction

Recently 'noise-induced phase transition' has attracted many worker's interest. Mostly, their attention is directed to the influence of the external noise on the stationary solution of the Fokker-Planck equation, and it has been shown that above a threshold value of the intensity of the external noise, the shape of the stationary solution suffers alteration.

In this paper we consider the dynamical aspects of noise-induced phase transition. We want to find whether the critical slowing down occurs or not near the noise-induced phase transition. We solve the moment equations corresponding to the Langevin equation under consideration with the aid of the method of asymptotic iteration (MAI) developed in the previous paper. We pay attention only to the first moment, because the determination of its temporal development is sufficient to know whether the critical slowing down occurs or not.

Now the stochastic differential equation (SDE) for our problem is given as

$$dX_t = (aX_t - X_t^{1+\rho})dt + X_t^{\rho}dB_t,$$ \hspace{1cm} (1.1)

where $B_t$ is the Wiener process with $\langle dB_t \rangle = 0$ and $\langle dB_t dB_t \rangle = \sigma^2 dt$. Here $\rho$ is a positive integer. It is to be noted that in the cases $\rho = 1$ and $\rho = 2$, SDE(1.1) represents the Verhulst model and the kinetic Ising model with a multiplicative noise respectively. As for the noise term we adopt the Stratonovich interpretation. According to the theorem of Wong and Zakai, the SDE should be interpreted as a Stratonovich equation as long as the limit of a realistic noise can be considered as the white noise. Moreover Kabashima et al. have shown that their experimental results are in good agreement with the theoretical results based on the Stratonovich interpretation. They found a noise-
induced phase transition in a degenerate parametric oscillator which was pumped by a random current simultaneously supplied with a sinusoidal current. These are the reasons why the Stratonovich interpretation is adopted in this paper.

The Fokker-Planck equation (FPE) corresponding to SDE (1.1) is written as

\[ \frac{\partial}{\partial t} P(X, t) = -\frac{\partial}{\partial X} \left\{ \left( a + \frac{\sigma^2}{2} \right) X - X^2 \right\} P(X, t), \]  

(1.2)

where \( P(X, t) \) is the probability density function. This FPE was solved by Schenzle and Brand as the eigenvalue problem. Their work is very splendid. However, we think that it is very complicated and lengthy to obtain the compact descriptions of moments with the aid of eigenfunctions, because the contribution of the continuous part of the eigenvalues is very complicated. And so we think that if we are only interested in the temporal development of the moments, we had better solve directly, the moment equations corresponding to the SDE.

In the next section, we treat the case \( \rho = 1 \) in (1.1) and give the exact temporal behavior of the first moment to discuss whether the critical slowing down occurs in the Verhulst model or not. In §3 we consider the general case with arbitrary \( \rho \). Section 4 is devoted to the concluding remarks.

§2. The Verhulst model with the multiplicative noise

The SDE (1.1) for \( \rho = 1 \), that is, the SDE

\[ dX_t = (aX_t - X_t^2)dt + X_t dB_t, \]  

(2.1)

corresponds to the Verhulst model with the multiplicative noise. The Verhulst model is one of the models in population dynamics. It expresses the temporal behavior of the number of individuals. Originally this model is the following phenomenological equation:

\[ \frac{d}{dt} x = Lx - x^2, \]  

(2.2)

where \( L \) is the intrinsic rate of increase and \(-x^2\) has its origin in a limited food supply. Here we consider a model for which \( L \) is a random variable in order to take account of the stochastic nature of the influence of the environment. Assuming the Gaussian white noise with the mean value \( a \) and the variance \( \sigma^2 \), we get the SDE (2.1). It is to be noted that we only consider the case \( a > 0 \). The FPE corresponding to the SDE (2.1) is given by

\[ \frac{\partial}{\partial t} P(X, t) = -\frac{\partial}{\partial X} \left\{ \left( a + \frac{\sigma^2}{2} \right) X - X^2 \right\} P(X, t). \]  

(2.3)
2.1. The stationary solution of (2·3)

To get the stationary solution $P_{st}(X)$ of (2·3) with the natural boundary condition $\lim_{X \to \pm \infty} P_{st}(X) = 0$, we need only to solve the equation

$$\left\{ \left( a + \frac{\sigma^2}{2} \right) X - X^2 \right\} P_{st}(X) = \frac{\sigma^2}{2} \frac{d}{dX} X^2 P_{st}(X).$$

(2·4)

The solution is easily found as

$$P_{st}(X) = \frac{X^{2a/\sigma^2-1}}{(\sigma^2/2)^{a/\sigma^2} \Gamma(2a/\sigma^2)} \exp \left( -\frac{2}{\sigma^2} X \right).$$

(2·5)

The extremum point $X_0$ of $P_{st}(X)$ is given by

$$X_0 = a - \sigma^2/2 \quad \text{for} \quad a > \sigma^2/2,$$

(2·6a)

$$X_0 = 0 \quad \text{for} \quad 0 < a < \sigma^2/2.$$  

(2·6b)

We find that the shape of stationary solution changes at $a = \sigma^2/2$. This is the noise-induced phase transition. Such discussion of the stationary state was made by Horsthemke and Malek-Mansour.\(^1\) However, as mentioned in § 1, we want to investigate whether the critical slowing down occurs or not near the noise-induced phase transition point.

2.2. The dynamical aspects

We solve the moment equations by MAI developed in the previous paper.\(^4\)

The moments are defined by

$$a_n(t) \equiv \int_0^\infty dX \cdot X^n P(X, t), \quad n = 1, 2, \ldots.$$  

(2·7)

The moment equations are given by

$$\frac{d}{dt} a_n(t) = n \left( a + \frac{1}{2} n \sigma^2 \right) a_n(t) - n a_{n+1}(t), \quad n = 1, 2, \ldots.$$  

(2·8)

They are transformed as

$$\frac{d}{dt} c_n(t) = \frac{1}{2} n^2 \sigma^2 c_n(t) - n e^{\sigma t} c_{n+1}(t), \quad n = 1, 2, \ldots,$$

(2·9)

$$c_n(t) = e^{-\sigma t} a_n(t).$$

We define $x(t)$ as the solution of the deterministic equation

$$\frac{d}{dt} x = ax - x^2.$$  

(2·10)

We also define $y(t)$ and $f_n(t)$ by
Dynamics of the Noise-Induced Phase Transition

$x(t) = e^{a(t)}y(t),$
$f_n(t) = c(t) - y^n(t).$

The differential equations (2·9) can be transformed into the following integral equations:

$$f_n(t) = \int_0^t ds e^{(1/2)\sigma y^n(t-s)} \left[ -ne^{a(t)}f_n(s) + \frac{1}{2} n^2 \sigma^2 y^n(s) \right].$$

(2·11)

It is to be noted that (2·11) does not have the term $e^{a(t)}f_n(0)$. This is because the case where the initial probability density function is given by $\delta(X-x(0))$ is considered here and in this case $f_n(0)$ ($n=1, 2, \cdots$) are found to be zero.

The first step of MAI is to solve (2·11) by iteration. Putting $f_n^{(1)}(t) (n=1, 2, \cdots; k=0, 1, 2, \cdots)$ for the $k$-th step terms and starting from $f_0^{(1)}(t)=0$, we have

$$f_n^{(k+1)}(t) = \int_0^t ds e^{(1/2)\sigma y^n(t-s)} \left[ -ne^{a(t)}f_n^{(1)}(s) + \frac{1}{2} n^2 \sigma^2 y^n(s) \right].$$

(2·12)

As mentioned in the introduction, we give only the temporal behavior of the first moment here. Needless to say, we can get the higher moments in a similar way.

Several beginning terms for $f_1(t)$, that is, $f_1^{(1)}(t)$, $f_1^{(2)}(t)$ and $f_1^{(3)}(t)$ are given as follows:

$$f_1^{(1)}(t) = b_1^{(1)}(t) = \frac{\sigma^2}{2} \int_0^t ds e^{(1/2)\sigma y^n(t-s)} y(s),$$

(2·13a)

$$f_1^{(2)}(t) - f_1^{(1)}(t) = b_1^{(2)}(t) + b_1^{(3)}(t),$$

(2·13b)

$$f_1^{(3)}(t) - f_1^{(2)}(t) = b_1^{(3)}(t) + b_1^{(4)}(t) + b_1^{(5)}(t),$$

(2·13c)

where

$$b_1^{(1)}(t) = -\frac{\sigma^2}{2} \frac{2^2}{\alpha + 3\sigma^2/2} \int_0^t ds y^3(s) e^{a(t)} e^{(1/2)\sigma y^n(t-s)}.$$

$$b_1^{(2)}(t) = \frac{\sigma^2}{2} \frac{2^2}{\alpha + 3\sigma^2/2} \int_0^t ds y^3(s) e^{2a(t)} e^{(1/2)\sigma y^n(t-s)}.$$

$$b_1^{(3)}(t) = \frac{\sigma^2}{2} \frac{2^2}{\alpha + 3\sigma^2/2} \int_0^t ds y^3(s) e^{3a(t)} e^{(1/2)\sigma y^n(t-s)}.$$

$$b_1^{(4)}(t) = \frac{\sigma^2}{2} \frac{2^2}{\alpha + 3\sigma^2/2} \int_0^t ds y^3(s) e^{4a(t)} e^{(1/2)\sigma y^n(t-s)}.$$

$$b_1^{(5)}(t) = \frac{\sigma^2}{2} \frac{2^2}{\alpha + 3\sigma^2/2} \int_0^t ds y^3(s) e^{5a(t)} e^{(1/2)\sigma y^n(t-s)}.$$

In general, $b_1^{(n)}(t)$ defined by
are given by
\[ b_m^k(t) = (-1)^{m-1} \left( \frac{2}{\sigma^2} \right)^{m-2} \frac{\Gamma(m+1+2a/\sigma^2)}{\Gamma(2m+2a/\sigma^2)} \times \int_0^t ds e^{-\sigma t}(e^{\sigma_1 y(s)})^m e^{(m+2)/2 \sigma^2 (t-s)} \times \frac{k^2}{(k-m)!} \frac{(m)_{k-m}}{(2m+1+2a/\sigma^2)_{k-m}} \left( \frac{2}{\sigma^2} e^{\sigma y(s)} \right)^{k-m}, \]

\[ k=1, 2, \ldots; \quad m=1, 2, \ldots, k. \]

Here \( (a)_n \) means \( a(a+1)(a+2)\cdots(a+n-1) \) and \( (a)_0 = 1 \).

Then the formal solution of \( f_m^{(m)}(t) \) is given by
\[ f_m^{(m)}(t) = \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} b_m^k(t) = \sum_{m=1}^{\infty} \sum_{k=m}^{\infty} b_m^k(t). \quad (2.16) \]

The second step of MAI is the selection of \( b_m^k(t) \), which are necessary to derive the asymptotic temporal development of \( f_m^{(m)}(t) \). In considering the temporal development of \( f_m^{(m)}(t) - f_m^{(m-1)}(t) \), \( b_m^k(t) \) is more divergent than \( b_m^2(t) \) but we cannot ignore \( b_m^k(t) \), because \( b_m^k(t) \) is of the same order as \( b_m^2(t) \) in the limit \( t \to \infty \). Then \( f_m^{(m)}(t) - f_m^{(m-1)}(t) \) must be put equal to \( b_m^2(t) + b_m^3(t) \). Similarly in considering the temporal development of \( f_m^{(m)}(t) - f_m^{(m-2)}(t) \) \( (k=3, 4, \ldots) \) we can ignore none of \( b_m^k(t) \) \( (m=1, 2, \ldots, k) \) even if \( b_m^k(t) \) is the most divergent of all \( b_m^k(t) \) \( (m=1, 2, \ldots, k) \). That is, in this model, it is found that all of \( b_m^k(t) \) must be summed up. By good fortune we can obtain the summation of \( b_m^k(t) \) exactly. First, the summation \( \sum_{k=m}^{\infty} b_m^k(t) \) is easily given by
\[ \sum_{k=m}^{\infty} b_m^k(t) = \frac{1}{(m-1)!} \frac{d}{dz} z \frac{d}{dz} z^{m-1} \int_0^t ds \int_0^1 du \left( u^{-1} - u \right)^{m+2a/\sigma^2} e^{(m+1)z \sigma^2 (t-s)} y_m(s) \times \exp \left( \frac{2z}{\sigma^2} e^{\sigma y(s)} u \right) \Bigg|_{u=1}. \quad (2.17) \]

It is to be noted that we use the following identity: \( 10 \)
\[ \frac{\Gamma(m+1+2a/\sigma^2)}{\Gamma(2m+2a/\sigma^2)} \Gamma(m) F \left( m, 2m+1+2a/\sigma^2; \frac{2z}{\sigma^2} e^{\sigma y(s)} \right) = \int_0^1 du u^{-1} (1-u)^{m+2a/\sigma^2} \exp \left( \frac{2z}{\sigma^2} e^{\sigma y(s)} u \right), \]
where $F(a, b; z)$ is the confluent hypergeometric function. Second, to sum up all of $\sum_{m=1}^{\infty} b_m^2(t)$ from $m$ equal to one to $m$ equal to infinity we make use of the identity

$$e^{(w/\sqrt{2})(t-s)} = e^{-w^2 - 2\sqrt{2}(t-s)w}/\sqrt{2\pi}.\int_{-\infty}^{\infty} du \exp(-u^2 + m\sqrt{2}\sigma^2(t-s)u).$$

Then we can get $f^{(m)}(t)$ as follows:

$$f^{(m)}(t) = \int_0^\infty \int_0^1 \int_{-\infty}^{\infty} dw \exp(-w^2 - m\sqrt{2}\sigma^2(t-s)w)$$

$$\times \left[ \frac{2a}{\sigma^2} e^{as} y(s) u \right] (1-u)^{1/2+2a/s^2}$$

$$\times \left( 1 + a/\sigma^2 - \frac{2a}{\sigma^2} e^{as} y(s) u (1-u) e^{2\sigma^2(t-s)w} \right)$$

$$\times \exp \left( -\frac{2a}{\sigma^2} e^{as} y(s) u (1-u) e^{2\sigma^2(t-s)w} \right). \quad (2.18)$$

Now it is to be noted that the first moment is given by $a_1(t) = x(t) + e^{at} f^{(m)}(t)$. The form of $f^{(m)}(t)$ given by (2.18) is too complicated to decide whether the critical slowing down occurs or not by direct inspection. Hence we calculate (2.18) numerically. We treat the case $x(t) = a = \text{const.}$, that is, $y(t) = ae^{-at}$. In this case we can integrate (2.17) with respect to $s$ and rewrite (2.17) as

$$\sum_{m=1}^{\infty} b_m^2(t) = (-1)^{m-1} e^{-at} \left( \frac{2a}{\sigma^2} \right)^{m-1} \frac{d^m}{dz^m} (2m + 2a/\sigma^2) \frac{d}{dz} z^m$$

$$\times \int_0^1 du \left( 1 - u \right)^{m+1} \left[ \frac{2a}{\sigma^2} z u \right] \exp \left( \frac{2a}{\sigma^2} z u (1-u) \right). \quad (2.19)$$

Then (2.18) can be rewritten as

$$f^{(m)}(t) = \frac{a}{\sqrt{\pi}} \frac{e^{-at}}{\sigma^2} \frac{d}{dz} \int_0^1 du \int_{-\infty}^{\infty} dw \exp(-w^2 - (2a/\sigma^2) \exp(2a/\sigma^2 u(1-u))$$

$$\times \left[ (2zu(1-u)) e^{as} e^{2\sigma^2 t} - 1 \right] \exp \left( -ze^{-at} e^{2\sigma^2 t} \right) \frac{2a}{\sigma^2} z u (1-u) \right]_{z=1}. \quad (2.20)$$

For $\sigma^2 \ll 2a$ and $t \gg 1$, the behavior of $e^{at} f^{(m)}(t)$ turns out to be proportional to $e^{-x(t-\sigma^2/2)}$ as shown in Fig. 1. Recently Shinozaki studied the same model as (2.1) in the limit $\sigma^2 \to 0$ and $t \to \infty$ for $\sigma^2 t$ fixed, with the aid of the perturbational method up to the first order of $\sigma^2$. He obtained the form of the correlation function as...
He concluded from this form that the critical slowing down occurs at $\alpha = \sigma^2/2$. We, however, think that his conclusion cannot be supported, because his result is correct only for the case $\sigma^2 > \alpha$ and cannot be used for the case where the order of $\sigma^2/2\alpha$ is unity.

The results of our numerical calculation show that the critical slowing down does not occur at $\alpha = \sigma^2/2$. As easily seen from Fig. 2, the larger $\sigma^2/2\alpha$ becomes, the slower and the more complicated the temporal behavior of $e^{\sigma t}f^{(n)}(t)$ becomes.

The discrete eigenvalues for the FPE (2.3) derived by Schenzle and Brand are given by

$$\lambda_n = n\sigma^2(\alpha/\sigma^2 - n/2), \quad \alpha/\sigma^2 \geq n.$$  

From this result we must not conclude that $\lambda_1 \to 0$ as $\sigma^2/2\alpha \to 1$ and then the critical slowing down occurs at $2\alpha = \sigma^2$ because there is no discrete eigenvalue for $\sigma^2/\alpha > 1$. That is, for $\sigma^2/\alpha > 1$, only the continuous branch exists. It makes the temporal behavior complicated. Of course, also in the case $\sigma^2/\alpha < 1$, there exists a continuous branch. And so, in the case where $\sigma^2/\alpha$ is nearly equal to 1 or larger than 1, the temporal behavior of $e^{\sigma t}f^{(n)}(t)$ turns out to be complicated. However, in the case where $\sigma^2/\alpha \ll 1$ and there are many discrete eigenvalues, the continuous part is overwhelmed by the discrete part in the limit $t \to \infty$, and then the temporal behavior tends to the exponential type.

In conclusion we want to say that the critical slowing down does not occur near the noise-induced phase transition in the Verhulst model with the multiplicative noise.

§ 3. The solution of the SDE (1·1)

In this section we treat the SDE (1·1) with arbitrary $\rho$. With the same notations as in the preceding section, we get the following integral equations:

$$f_{n+1}^{(n+1)}(t) = \int_0^t ds e^{(n+2)\sigma^2\alpha t - s} \left( -ne^{\rho s}f_{n+1}^{(n)}(s) + \frac{n^2}{2} \sigma^2 s^n(s) \right),$$  

for $x(t) = \alpha + \sigma \xi(t)$. The critical slowing down occurs at $\alpha = \sigma^2/2$. We, however, think that his conclusion cannot be supported, because his result is correct only for the case $\sigma^2 > \alpha$ and cannot be used for the case where the order of $\sigma^2/2\alpha$ is unity. The results of our numerical calculation show that the critical slowing down does not occur at $\alpha = \sigma^2/2$. As easily seen from Fig. 2, the larger $\sigma^2/2\alpha$ becomes, the slower and the more complicated the temporal behavior of $e^{\sigma t}f^{(n)}(t)$ becomes.

The discrete eigenvalues for the FPE (2·3) derived by Schenzle and Brand are given by

$$\lambda_n = n\sigma^2(\alpha/\sigma^2 - n/2), \quad \alpha/\sigma^2 \geq n.$$  

From this result we must not conclude that $\lambda_1 \to 0$ as $\sigma^2/2\alpha \to 1$ and then the critical slowing down occurs at $2\alpha = \sigma^2$ because there is no discrete eigenvalue for $\sigma^2/\alpha > 1$. That is, for $\sigma^2/\alpha > 1$, only the continuous branch exists. It makes the temporal behavior complicated. Of course, also in the case $\sigma^2/\alpha < 1$, there exists a continuous branch. And so, in the case where $\sigma^2/\alpha$ is nearly equal to 1 or larger than 1, the temporal behavior of $e^{\sigma t}f^{(n)}(t)$ turns out to be complicated. However, in the case where $\sigma^2/\alpha \ll 1$ and there are many discrete eigenvalues, the continuous part is overwhelmed by the discrete part in the limit $t \to \infty$, and then the temporal behavior tends to the exponential type.

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Dynamics of the Noise-Induced Phase Transition

Fig. 2. The temporal development of $e^{|\varphi|^2}/(\alpha/(\alpha-1))$: (a) $\sigma/2\alpha=.1$, (b) $\sigma/2\alpha=.3$, (c) $\sigma/2\alpha=5$, (d) $\sigma/2\alpha=9$, (e) $\sigma/2\alpha=.99$, (f) $\sigma/2\alpha=1.1$.

$n=1, 2, \cdots; \quad k=0, 1, 2, \cdots$

where $y(t)$ is the solution of the equation

$$\frac{d}{dt}y = -e^{\sigma^2}y + \varphi^2 \beta.$$ \hfill (3.2)

The first moment $a_1(t)$ is given by

$$a_1(t) = e^{\sigma^2}(y(t) + \varphi^2(t)).$$ \hfill (3.3)

Starting from $f_n^{(0)}(t)=0$, we can get the several beginning terms for $f_i(t)$ as follows:

$$f_l^{(1)}(t) = b_l^{(1)}(t) = \frac{\sigma}{2} \int_0^t ds e^{\sigma^2 2(t-s)}y(s),$$ \hfill (3.4a)

$$f_l^{(2)}(t) = b_l^{(2)}(t) + b_l^{(1)}(t),$$ \hfill (3.4b)

$$f_l^{(3)}(t) = b_l^{(3)}(t) + b_l^{(2)}(t) + b_l^{(1)}(t).$$ \hfill (3.4c)
Here we define \( b(k,t)(t) \) by the relations

\[
 f^{(k)}(t) = f^{(k-1)}(t) \equiv \sum_{m=1}^{k} b(k,t)(t).
\]

Generally \( b(k,t)(t) \) are given by

\[
b(k,t)(t) = \left( -1 \right)^{m-1} \frac{\sigma^2}{2} \left( \frac{1}{\rho} \right)^{k-1} \left( \frac{1}{1 + (k-1)\rho} \right)^{m-1} \left( \frac{1}{1 + (k+1)\rho} \right)^{k-m-1} \times \frac{\Gamma(2a/\rho + 2/\rho + m-1)}{\Gamma(2a/\rho + 2/\rho + m-2)}
\]

\[
 \times \int_0^t ds e^{-s}(e^{\sigma y(s)})^{1+(m-1)\rho} e^{\sigma y(s)} \frac{(e^{\sigma y(s)})^{k-m} + (e^{\sigma y(s)})^{k-m+1}}{(k-m)! (k-m+1)!}.
\]

Then the straightforward calculation gives the following results:

\[
 \sum_{k=m}^{\infty} b(k,t)(t) = (-1)^{m-1} \frac{\sigma^2}{2} \left( \frac{2}{\rho \sigma^2} \right)^{m-1} \times \frac{(2a/\rho + 2/\rho)_{m-1} + 2(m-1)(2a/\rho + 2/\rho + 1)_{m-2}}{(m-1)! (2a/\rho + 1/\rho + 1)_{m-1}}.
\]
Dynamics of the Noise-Induced Phase Transition

\[ \times \frac{\Gamma(2a/\rho \sigma^2 + 2/\rho + 1)}{\Gamma(2a/\rho \sigma^2 + 1/\rho + 1) \Gamma(1/\rho)} \frac{d}{dz} \frac{d}{dz} z^{1-(m-1)\rho} \]

\[ \times \int_0^t ds \int_0^t du y(s)(e^{at}y(s))^m \exp\left( \frac{1}{2} (1 + (m - 1)\rho) \sigma^2 (t - s) \right) \]

\[ \times \exp\left\{ \frac{2}{\rho \sigma^2} (ze^{at}y(s))u \right\} u^{-2+1/\rho}(1 - u)^{2a/\rho \sigma^2 + 1 - 1/\rho} \}

And \( f^{(m)}(t) \) can be written as

\[ f^{(m)}(t) = C \frac{d}{dz} z \int_0^t ds \int_0^t du B \]

\[ \times \left\{ F(2a/\rho \sigma^2 + 2/\rho, 2a/\rho \sigma^2 + 1/\rho + 1; -2A/\rho \sigma^2) \right. \]

\[ - \frac{4A}{2a + (1 + \rho) \sigma^2} \]

\[ \times \left. F(2a/\rho \sigma^2 + 2/\rho + 1, 2a/\rho \sigma^2 + 1/\rho + 2; -2A/\rho \sigma^2) \right\} z = 1, \]

where

\[ A = (ze^{at}e^{s\sigma \sqrt{2(t - s)}} y(s))^u (1 - u), \]

\[ B = \exp\left( -w^2 + aw \sqrt{2(t - s)} \right) \]

\[ + \frac{2}{\rho \sigma^2} (ze^{at}y(s))^u y(s) u^{1-(m-1)\rho}(1 - u)^{2a/\rho \sigma^2 + 1 - 1/\rho}. \]

\[ C = \frac{\sigma^2}{2\sqrt{\pi}} \frac{\Gamma(2a/\rho \sigma^2 + 2/\rho + 1)}{\Gamma(2a/\rho \sigma^2 + 1/\rho + 1) \Gamma(1/\rho)}. \]

Here \( F(a, b; z) \) is the confluent hypergeometric function.

We can easily find that Eq. (3.5) coincides with Eq. (2.18) in the case \( \rho = 1 \) owing to the identity

\[ F(a, a; z) = e^z. \]

As the first moment is written by Eq. (3.3) we have obtained the exact temporal development of it in the physical systems which are expressed by SDE (1.1). In the case \( \rho = 2 \), SDE (1.1) corresponds to the kinetic Ising model with the multiplicative noise. In this paper we do not enter into the detailed numerical calculation of (3.5). Here we let ourselves be contented by emphasizing that Eq. (3.4) is the exact expression for the solution of (3.1).
§ 4. Concluding remarks

In § 2 we reached the conclusion that the critical slowing down does not occur in the Verhulst model with the multiplicative noise. As described there, the larger \( \sigma^2/2a \) becomes, the longer the time for approaching the stationary state becomes. We think that this may be related to the fact that the mean value of the stationary probability density function, which is given by \( \int_0^\infty dx \cdot XP_a(x) = a \), is not dependent on the magnitude of noise. Namely, in this phase transition, the stationary value of the first moment does not change at the transition point, so the critical slowing down does not occur as far as the temporal development of the first moment is concerned. Here we note that Suzuki\(^{12}\) considered the temporal development of \( \int_0^\infty dx \cdot X^{-1}P(X, t) \) and got the result that the critical slowing down occurs.

Our initial distribution can be made to be more general. We think that even if the initial probability density function is not the \( \delta \)-function, the time for approaching the stationary state becomes larger as the magnitude of noise becomes larger. From the results of our numerical calculation we can conclude that the critical slowing down does not occur near the noise-induced phase transition point in the Verhulst model with the multiplicative noise.

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