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Radical Root Structure of Conformal Algebra

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We construct radical roots of the elements of the conformal algebra in two dimensions and then study the exotic graded algebraic structure satisfied by them. This algebraic structure consists of a graded version of the commutator and the anti-commutator and inherits the characteristic feature of the super conformal algebra. Thus it suggests a possibility of the extension of the concept of supersymmetry.

§ 1. Introduction

We shall expose here an exotic algebraic structure V_M represented on fields on a Riemann surface. V_M is a kind of an M-th root structure of the conformal algebra, which is the infinite dimensional Lie algebra generated by infinitesimal conformal transformations. Such a concept is no doubt mathematically interesting. Besides, it seems potentially useful in $2d$ conformal field theory.[1] The special case V_2 is, for example, the well-known super conformal algebra[2),3)] with no central charge. Hence, naively we expect that V_M generates a generalization of the super conformal symmetry. Indeed, in a concrete lagrangian model one can construct the currents which generate the infinitesimal transformations corresponding to the elements of V_M and the lagrangian is invariant under these infinitesimal transformations.[4]

However, at present our understanding of the role of the algebra V_M is quite limited. For example, one main problem in applying it to the conformal field theory is that one cannot realize the algebra V_M as a current algebra itself. We leave the deeper understanding of the algebra V_M as a future problem and here we will devote ourselves in presenting this unexpected algebraic structure.

Before giving the definition of V_M, we shall explain the key idea of the construction in a much simpler algebraic system as an illustration. Consider an abelian group G. One can canonically construct the M-th root structure of G as follows. Let I_M be the unit $M\times M$ matrix. We will identify $a\in G$ as $a\cdot I_M$, namely, a G-valued $M\times M$ matrix. Now it is trivial that the element

$$b=\left[\begin{array}{c|ccc} & 1 & & \\ & & \ddots & \\ & & & 1 \\ \hline a & & & \end{array}\right] \tag{1}$$

is the M-th root of a, where and in what follows the blank in matrices denotes zero. By adding the M-th roots of the elements of G and their products, one obtains the M-th root structure of G, which is no longer abelian. What we want to construct is not the M-th root of a group but the one of a Lie algebra. This makes the situation rather complicated. Especially, the grading and the graded product will be introduced in order to close the algebra.

§ 2. Radical root structure of the conformal algebra

Let ∂ be the formal differential operator with respect to the variable z. For each integer m, consider the operator $L_m = z^{m+1}\partial$ acting on the space $F(z)$ of all the formal Laurent series of z. The vector space V_1 spanned by the basis L_m has the following Lie algebraic structure

$$[L_m, L_n] = -(m-n)L_{m+n} \tag{2}$$

known as the conformal algebra, or also known as the Witt algebra or the Virasoro algebra with no central charge. For our purpose, it is useful to define the more general operators

$$L_m(h) = z^{m+1}\partial + h(\partial z^{m+1}) \tag{3}$$

for a fixed number h. These operators also satisfy the same Lie algebra, namely,

$$[L_m(h), L_n(h)] = -(m-n)L_{m+n}(h)\,. \tag{4}$$

Suppose M is a fixed positive integer and $k=1, \cdots, M$. We define D^k and $E_k(h)$ to be operators acting on the space $\oplus^M F(z^{1/M})$, the direct sum of M copies of $F(z^{1/M})$, as

$$D^k = \left[\begin{array}{ccc|ccc} & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \\ \hline \partial & & & & & \\ & \ddots & & & & \\ & & \partial & & & \end{array}\right]\begin{array}{l} \\ \\ \\ \left.\begin{array}{l} \\ \\ \\ \end{array}\right\} k \end{array},$$

$$E_k(h) = \frac{1}{k}\left[\begin{array}{cccc|c} & & & & \\ & & & & \\ & & & & \\ \hline Mh & & & & \\ & Mh+1 & & & \\ & & \ddots & & \\ & & & Mh+k-1 & \end{array}\right]\begin{array}{l} \\ \\ \\ \left.\begin{array}{l} \\ \\ \\ \\ \end{array}\right\} k \end{array}. \tag{5}$$

Then the operator $R_m^{(k)}(h)$ on $\oplus^M F(z^{1/M})$ is defined by

$$R_m^{(k)}(h)=z^{m+k/M}D^k+(\partial z^{m+k/M})E_k(h)\,, \tag{6}$$

where $k=1, \cdots, M$. In particular,

$$R_m^{(M)}(h)=\begin{bmatrix} L_m(h) & & & \\ & L_m\left(h+\frac{1}{M}\right) & & \\ & & \ddots & \\ & & & L_m\left(h+\frac{M-1}{M}\right) \end{bmatrix}. \tag{7}$$

For the moment, we do not specify the range in which the suffix m attains value. One may consider it, for example, as $\boldsymbol{Z}$, $\boldsymbol{Q}$, $\boldsymbol{R}$, or $(1/N)\boldsymbol{Z}$ for some positive integer N. Next let H_k be the operator on $\oplus^M F(z^{1/M})$ given by

$$H_k=\left[\begin{array}{c:c} & \\ \hdashline & \\ 1 & \end{array}\right]\begin{array}{l} \\ \left.\vphantom{\begin{array}{c}1\\1\end{array}}\right\} k \end{array}. \tag{8}$$

The following formulas are useful, where $D=D^1$.

i) For $k=2, \cdots, M$, $\quad Dz^nE_{k-1}(h)=z^n\left(\dfrac{k}{k-1}E_k(h)-\dfrac{Mh+k-1}{k-1}H_k\right)$,

$$E_1(h)z^nE_{k-1}(h)=0\,.$$

ii) For $k=1, \cdots, M$, $\quad Dz^nD^{k-1}=z^nD^k+(\partial z^n)H_k$,

$$E_1(h)z^nD^{k-1}=z^nMhH_k\,. \tag{9}$$

Let $\{a_1, \cdots, a_k\}$ denote $(1/k!)\sum_\sigma a_{\sigma(1)}\cdots a_{\sigma(k)}$, namely, the normalized totally symmetric product, where the summation is carried over the set of all the permutations of $\{1, \cdots, k\}$. Then the fundamental fact is

Proposition 1 For $k=2, \cdots, M$,

$$\{R_{m_1}^{(1)}(h), \cdots, R_{m_k}^{(1)}(h)\}=R_{m_1+\cdots+m_k}^{(k)}(h)\,. \tag{10}$$

proof As the induction hypothesis, we assume that the relation holds for $k-1$. Let m be the sum $\sum_{j=1}^k m_j$, then

$$\{R_{m_1}^{(k)}(h), \cdots, R_{m_k}^{(1)}(h)\}=\frac{1}{k!}\sum_{j=1}^{k}R_{mj}^{(1)}(h)(k-1)!R_{m-m_j}^{(k-1)}(h)\,.$$

Using Eq. (9),

$$R^{(1)}_{m_j}(h)R^{(k-1)}_{m-m_j}(h)=z^{m+k/M}D^k+z^{m_j+1/M}(\partial z^{m-m_j+(k-1)/M})H_k+z^{m_j+1/M}(\partial z^{m-m_j+(k-1)/M})$$

$$\times\left(\frac{k}{k-1}E_k(h)-\frac{Mh+k-1}{k-1}H_k\right)+(\partial z^{m_j+1/M})z^{m-m_j+(k-1)/M}MhH_k\,.$$

Thus,

$$\{R^{(1)}_{m_1}(h),\cdots,R^{(1)}_{m_k}(h)\}=\frac{1}{k}\sum_{j=1}^{k}\left\{z^{m+k/M}D^k+\left(m-m_j+\frac{k-1}{M}\right)\frac{k}{k-1}z^{m+k/M-1}E_k(h)\right.$$

$$\left.+(km_j-m)Mhz^{m+k/M-1}H_k\right\}$$

$$=R^{(k)}_{m_1+\cdots+m_k}(h)\,. \qquad \text{q.e.d.}$$

Proposition 1 shows that any element $R_m{}^{(k)}(h)$ can be expressed in the symmetric product of k elements $R^{(1)}_{m_i}(h)$ with $m=\sum_{i=1}^{k}m_i$. We call such an expression a decomposition of $R_m{}^{(k)}(h)$. A decomposition of $R_m{}^{(k)}(h)$ is not unique. Notice that, as a special case, $R^{(k)}_{mk}(h)$ has its k-th root $R_m{}^{(1)}(h)$.

After a straightforward calculation, one obtains another important commutation relation.

$$[R_m{}^{(M)}(h),R_n{}^{(k)}(h)]=-\left(\frac{k}{M}m-n\right)R^{(k)}_{m+n}(h)\,. \tag{11}$$

For a fixed h, let $V_M{}^k(h)$ denote the space spanned by the set of basis $\{R_m{}^k(h)\}_m$ and let $V_M(h)$ denote the direct sum $\oplus_{k=1}^{M}V_M{}^k(h)$, where $k=1,\cdots,M$ gives the grading of the space. The above relations are independent h and, as well shall see, essentially determine the algebraic structure of $V_M(h)$. Hence we will suppress h and will simply write $R_m{}^{(k)}$, V_M and so on. It should be remarked that there are no commutators between $R_m{}^{(k)}$'s which close in V_M except for Eq. (11).

The next step is to define a graded product $\{\,,\}:V_M\times V_M\to V_M$, which is a bilinear map satisfying the following property:

$$\text{If}\quad a\in V_M{}^k,\quad b\in V_M{}^j,\quad\text{then}\quad \{a,b\}\in\begin{cases}V_M{}^{k+j} & \text{if}\quad k+j\leq M\,,\\ V_M{}^{k+j-M} & \text{if}\quad k+j>M\,.\end{cases} \tag{12}$$

Suppose that $k+j\leq M$. Let m, n be given numbers and $m=\sum_{i=1}^{k}m_i$, $n=\sum_{i=k+1}^{k+j}m_i$ be their arbitrary decompositions. We shall define the *bracket product* $\{R_m{}^{(k)},R_n{}^{(j)}\}$ as

$$\frac{1}{(k+j)!}\sum_{\substack{\sigma:\text{ permutation}\\ \text{of }\{i\}_{i=1}^{k+j}}}R^{(k)}_{m_{\sigma(1)}+\cdots+m_{\sigma(k)}}R^{(j)}_{m_{\sigma(k+1)}+\cdots+m_{\sigma(k+j)}}\,. \tag{13}$$

Then using the decompositions of $R_m{}^{(k)}$ and $R_n{}^{(j)}$, it is not difficult to show that

$$\{R_m{}^{(k)},R_n{}^{(j)}\}=R^{(k+j)}_{m+n}\,. \tag{14}$$

Equation (14) also shows that the definition of the bracket product does not depend on the choice of the decompositions $m=\sum_{i=1}^{k}m_i$, $n=\sum_{i=k+1}^{k+j}m_i$.

As an illustration, we will show an example of $k=j=2$, $m=4=1+3$, $n=6=2+4$ and $M\geq 4$. In this case,

$$\{R_{1+3}^{(2)}, R_{2+4}^{(2)}\} = \frac{2!2!}{4!}(R_4{}^{(2)}R_6{}^{(2)} + R_3{}^{(2)}R_7{}^{(2)} + R_5{}^{(2)}R_5{}^{(2)} + R_5{}^{(2)}R_5{}^{(2)}$$
$$+ R_7{}^{(2)}R_3{}^{(2)} + R_6{}^{(2)}R_4{}^{(2)})$$
$$= \frac{1}{4!}(\{R_1{}^{(1)}, R_3{}^{(1)}\}\{R_2{}^{(1)}, R_4{}^{(1)}\} + \{R_1{}^{(1)}, R_2{}^{(1)}\}\{R_3{}^{(1)}, R_4{}^{(1)}\}$$
$$+ \{R_1{}^{(1)}, R_4{}^{(1)}\}\{R_3{}^{(1)}, R_2{}^{(1)}\} + \{R_3{}^{(1)}, R_2{}^{(1)}\}\{R_1{}^{(1)}, R_4{}^{(1)}\}$$
$$+ \{R_3{}^{(1)}, R_4{}^{(1)}\}\{R_1{}^{(1)}, R_2{}^{(1)}\} + \{R_2{}^{(1)}, R_4{}^{(1)}\}\{R_1{}^{(1)}, R_3{}^{(1)}\})$$
$$= R_{10}^{(4)} . \tag{15}$$

Our definition of the bracket product is clearly a generalization of the anticommutator. However, at first sight it might look unusual that not only $R_4{}^{(2)}$, $R_6{}^{(2)}$ but also other operators $R_5{}^{(2)}$, $R_3{}^{(2)}$ appear in the definition of the product for $R_4{}^{(2)}$ and $R_6{}^{(2)}$. A proper interpretation is that this algebraic operation is defined not through $R_m{}^{(k)}$ itself but through the *decomposition* of it. This point is indeed a characteristic feature of this graded algebraic structure.

In the case $k+j>M$, the situation is more complicated but the same feature will appear. The only natural operation one expects is perhaps to rearrange $R_m{}^{(k)}$ and $R_n{}^{(j)}$ into two elements $a \in V_M{}^M$, $b \in V_M{}^{k+j-M}$, and then to take the commutator $[a, b]$. To get a further insight, let us consider the well-known case $M=2$, namely, the super conformal algebra or the Ramond-Neveu-Schwarz algebra. What we are interested in is the case $k=1$, $j=2$. Let us adopt the conventional notation $G_m = R_m{}^{(1)}$, $L_m = R_m{}^{(2)}$. Consider the commutator

$$[G_m, L_m] = \left(\frac{1}{2}n - m\right)G_{n+m} . \tag{16}$$

Then substituting the expression $L_n = \{G_{n_1}, G_{n_2}\}$ with $n = n_1 + n_2$ into the l.h.s. of Eq. (16) and using the $\boldsymbol{Z}_2$-graded Jacobi identity

$$[G_m, \{G_{n_1}, G_{n_2}\}] + [G_{n_1}, \{G_{n_2}, G_m\}] + [G_{n_2}, \{G_m, G_{n_1}\}] = 0 , \tag{17}$$

one gets another equivalent expression of the commutator in (16),

$$[\{G_m, G_{n_1}\}, G_{n_2}] + [\{G_m, G_{n_2}\}, G_{n_1}] = \left(\frac{1}{2}n - m\right)G_{n+m} . \tag{18}$$

Notice that the result does not depend on the choice of the decomposition $n = n_1 + n_2$. One interpretation of Eq. (18) is that one can define a closed operation by taking the decomposition of $L_n = \{G_{n_1}, G_{n_2}\}$, joining one part of the decomposition with G_m by the bracket product and then taking the commutator with the other part of the decomposition.

Now we return to the general cases. Suggested by the expression in Eq. (18), we shall *again* define the bracket product $\{R_m{}^{(k)}, R_n{}^{(j)}\}$ not by $R_n{}^{(j)}$ itself but by the decomposition $R_n{}^{(j)} = \{R_{n_1}^{(M-k)}, R_{n_2}^{(k+j-M)}\}$ with $n = n_1 + n_2$. To make the bracket product independent of the decomposition of n, it will turn out that the additional symmetrization of the decomposition of $R_n{}^{(j)}$ is necessary. Thus we are led to the following definition of the bracket product,

$$\frac{1}{k+j-M}\frac{1}{(j-1)!}\sum_{\substack{\sigma:\text{ permutation}\\ \text{of } \{i\}_{i=1}^{j}}}[\{R_m{}^{(k)}, R^{(M-k)}_{n_{\sigma(1)}+\cdots+n_{\sigma(M-k)}}\}, R^{(k+j-M)}_{n_{\sigma(M-k+1)}+\cdots+n_{\sigma(j)}}] \tag{19}$$

with $n=\sum_{i=1}^{j}n_i$. The numerical factor is just a convention but this choice will turn out to be convenient. In the case $k=M$, this braket product reduces to the commutator $[R_m{}^{(k)}, R_n{}^{(j)}]$ so that it is a graded version of the ordinary commutator. This bracket product obeys the very simple expression.

Proposition 2

$$[R_m{}^{(k)}, R_n{}^{(j)}\}=\begin{cases} R^{(k+j)}_{m+n} & \text{if} \quad k+j\leq M\,, \\ -\left(\dfrac{j}{M}m-\dfrac{k}{M}n\right)R^{(k+j-M)}_{m+n} & \text{if} \quad k+j>M\,. \end{cases} \tag{20}$$

proof We have only to verify the latter case. By definition, each term of the l.h.s. is

$$[R^{(M)}_{m+n_{\sigma(1)}+\cdots+n_{\sigma(M-k)}}, R^{(k+j-M)}_{n_{\sigma(M-k+1)}+\cdots+n_{\sigma(j)}}]=\left\{-\frac{k+j-M}{M}(m+\sum_{i=1}^{M-k}n_{\sigma(i)})+\sum_{i=M-k+1}^{j}n_{\sigma(i)}\right\} \times R^{(k+j-M)}_{m+n}\,.$$

Since for a fixed i,

$$\sum_{\substack{\sigma:\text{ permutation}\\ \text{of } \{i\}_{i=1}^{j}}} n_{\sigma(i)}=(j-1)!\sum_{i=1}^{j}n_i=(j-1)!n\,,$$

after some straightforward calculations, we get the desired result. q.e.d.

The proposition shows that the bracket product is independent of the choice of the decomposition of $R_n{}^{(j)}$. As a corollary of Proposition 2, the bracket product has the following (anti-)commutativity. In general cases, it gives a non-trivial identities of the $R_m{}^{(k)}$'s.

Let $a\in V_M{}^k$, $b\in V_M{}^j$, then

$$\{a, b\}\in\begin{cases}\{b, a\} & \text{if} \quad k+j\leq M\,, \\ -\{b, a\} & \text{if} \quad k+j>M\,. \end{cases} \tag{21}$$

Here we shall give some identities satified by the combinations of bracket products $\{\{a, b\}, c\}$. Generally, one cannot expect any simple relations from the definition of the bracket product except for the trivial cases. However, the results in Proposition 2 show that all $\{\{a, b\}, c\}$, $\{\{b, c\}, a\}$, $\{\{c, a\}, b\}$ coincide up to the coefficient. Thus, there are indeed infinite relations of the following type

$$u_1\{\{a, b\}, c\}+u_2\{\{b, c\}, a\}+u_3\{\{c, a\}, b\}=0\,. \tag{22}$$

Among them, we will point out some special identities which possess the symmetry between u_1, u_2 and u_3.

Let $a \in V_M^{k_1}$, $b \in V_M^{k_2}$, $c \in V_M^{k_3}$. Then

$$u_1\{\{a, b\}, c\} + u_2\{\{b, c\}, a\} + u_3\{\{c, a\}, b\} = 0\,, \tag{23}$$

if 1) $M < k_1 + k_2 + k_3 \leq 2M$, and

a) $k_1+k_2,\ k_2+k_3,\ k_3+k_1 \leq M$, $\quad u_1 = u_2 = u_3 = 1$,

b) $k_1+k_2 > M$; $k_2+k_3,\ k_3+k_1 \leq M$, $\quad u_1 = k_1+k_2+k_3,\ u_2 = k_2,\ u_3 = -k_1$,

c) $k_1+k_2,\ k_2+k_3 > M$; $k_3+k_1 \leq M$, $\quad u_1 = 1,\ u_2 = -1,\ u_3 = -1$,

d) $k_1+k_2,\ k_2+k_3,\ k_3+k_1 > M$, $\quad u_1 = k_3,\ u_2 = k_1,\ u_3 = k_2$,

2) $2M < k_1+k_2+k_3 \leq 3M$, $\quad u_1 = u_2 = u_3 = 1$.

So far, we do not care about the range of the suffix m. In the case V_2, we know that there are two specific choices of it, i.e., $\boldsymbol{Z}$ (Ramond sector) and $\boldsymbol{Z}+k/2$ (Neveu-Schwarz sector) for V_2^k. These choices are minimal including the original conformal abgebra. For V_M, there are M different such minimal ranges. Namely, for a fixed $s \in \{1/M, 2/M, \cdots, 1\}$, set $m \in \boldsymbol{Z}+sk$ for $R_m^{(k)}$. Then V_M spanned by these bases is closed under the bracket product.

§ 3. Extension to operators on a Riemann surface

The operators in V_M can be extended to the operators defined on a Riemann surface. First, for example, consider the operator $L_\xi(h) = \xi\partial + h(\partial\xi)$ for an arbitrary meromorphic field ξ of weight -1 on a Riemann surface S. Locally, $L_\xi(h)$ is expressed by a Laurent series $\sum a_m L_m(h)$. Clearly, $L_\xi(h)$ acts on the space of fields of weight h. In the same way, we wish to define the operator

$$R_\xi^{(k)}(h) = \xi D^k + (\partial\xi) E_k(h)\,. \tag{24}$$

The question is on what space $R_\xi^{(k)}(h)$ acts. Let Γ_h denote a certain class of fields of generally fractional weight h, which are locally expressed as a multi-valued function. Suppose ξ be a field of weight $-k/M$ locally expanded as $\sum_{m \in (1/N)Z} a_m z^m$ for some positive integer N. Then it is not difficult to see that $R_\xi^{(k)}(h)$ acts on a multiple

$$\begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_{M-1} \end{bmatrix}, \tag{25}$$

where $f_j \in \Gamma_{h+j/M}$. Namely, the action of $V_M(h)$ can be extended to the operation on the space of multiples of fields on a Riemann surface whose weights are $h, h+1/M, \cdots, h+(M-1)/M$. Thus again it is a natural generalization of the characterization of the super conformal transformations.

§ 4. Conclusion and discussion

We constructed radical roots of the elements of the conformal algebra and then presented an algebraic structure closed under the bracket product. It inherits some of the characteristic features of the super conformal algebra. First, the bracket product is a graded version of the commutator and the anti-commutator. Second, the action of V_M can be extended to the space of multiples of fields with different weights on a Riemann surface. So we suspect that they indicate an unknown symmetric structure which is a certain generalization of the super conformal symmetry. On the other hand, we lack the understanding of the true nature of this algebraic structure, especially, its role in quantum field theory. One of the important problems to be solved is to incorporate this algebraic structure in the operator method of $2d$ conformal field theory, and especially to clarify the quantum-statistical property of the currents which generate these infinitesimal transformations. Maybe it will be necessary to introduce a totally new type of statistics. Also the search for the geometry associated to V_M is an important problem to be investigated. Another direction of development is to apply our construction to affine Kac-Moody algebras. In fact, in an analogous way one can construct a radical root of the affine $U(1)$ algebra, which is the simplest case. It is interesting that the operators in this system might play an important role in constructing the operator formalism on a Riemann surface with an infinite genus following Matsuo's suggestion.[5]

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