Spherical Pendulum in Gravitational Experiments

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The fine structure of the trajectory of the Foucault pendulum with anisotropic suspension is investigated. In respect of non-Newtonian gravitational effects, a mathematical theory of this device is developed and an analytic formula for the precession of plane of swing as a function of the initial conditions and anisotropy of suspension is obtained. Effects of viscous friction, seismic noise and dynamical instability are investigated. The characteristics of the pendulum dynamics are compared with the results of existing and planned gravitational experiments.

§1. Introduction

A mathematical definition of the spherical pendulum as a mechanical device corresponds to a material point body which can undergo periodic motion on a spherical surface under the action of the gravitational force. The radius of this conceivable surface is defined by the length of the pendulum arm. A principal feature of the device is a degeneration of modes with different swing planes. From the physical point of view this means the pendulum must be completely isotropic. For a pendulum with a wire suspension, an isotropy of bending elastic properties of the wire is required; in the case of a rod suspension with some ball support at the top point, an isotropy of the support reaction for any direction of the plane of swing is assumed.

The completely isotropic spherical pendulum is usually referred to as the "Foucault pendulum", named after Jean Foucault, who used it in 1851 for the first demonstration of the Earth's rotation. In the linear approximation, a "Foucault precession", as a rotation of the pendulum swing plane with angular velocity \( \dot{\Omega} = \omega_0 \sin \lambda \) (\( \lambda \) is the pendulum latitude) has been easily interpreted in both the inertial reference frame fixed to remote stars (the law of angular momentum conservation) and in the non-inertial frame fixed to the Earth (the result of the Coriolis force action). The effect was very impressive for spectators, and seemed completely understandable.

However, subsequent study of the Foucault pendulum's behavior revealed its nontrivial dynamics caused by nonlinear properties of the device. Carefully taking into account nonlinearity results in a more complex character of precession, which is no longer monotonic; its period begins to depend on the pendulum amplitude, etc. In the first nonlinear approximation, the pendulum trajectory is not a "plane curve" and the pendulum end describes an ellipse (as projected on the ground), with the major axis undergoing precession.

One can find in the literature papers dealing with the problem of nonlinear...
analysis of the Foucault pendulum (see for example, Refs. 3-8)). In fact these studies have found nonlinear corrections to both pendulum and precession frequencies; also a more precise trajectory dynamics was derived.

Nevertheless, behavior of real devices went all over theoretical frames. A reason for this is the difference between experimental set-ups and the ideal model of the Foucault pendulum, first of all due to the presence of some anisotropy and asymmetry typical for any practical device. In addition there are always stochastic perturbations due to vibrations, friction, air currents, etc. For these reasons modern models of the pendulum are equipped with a very complex electro-mechanics compensating for perturbations in order to make the pendulum behavior closer to the ideal linear precession. The accuracy of the best models at present is with an error of several percent. 9)

In spite of its complexity, the spherical pendulum has been used for measurements of weak non-Newtonian gravitational anomalies. Here we would like to mention two kinds of such papers.

1. In an experiment by Allaise10) an attempt has been made to detect some anomalies in the precession of the spherical pendulum during a solar eclipse, appealing to the hypothetical opportunity of a "gravitational screen". According to his claim10) some "unusual acceleration" of the precession velocity occurred. Subsequent experiments (see for example, Refs. 11) and 12)) did not provide a reliable support for the "Allaise effect" having, however, left room for possible speculation open.

2. Braginsky, Polnarev and Thorne have proposed that the Foucault pendulum for a measurement of the gravimagnetic Lense-Thirring effect (a perturbation of the Earth's gravitational field due to rotation)13) considering it as a competitive variant of the project of "relativistic gyroscope on the Earth orbit".14),15) In their scheme a Foucault pendulum placed on the southern pole must perform an additional secular precession in the swing plane, induced by the "rotational gravitational field" of the Earth. The expected declination must be extremely small, smaller than the magnitude of Foucault effect by a factor of 10^{-10}.

Coming back to the Allaise experiments, we should remark that it is unlikely to observe larger values there also, remaining in the frame of conventional gravitational physics.

It is clear that measurements of such weak perturbations require a detailed theoretical analysis of the fine structure of the natural trajectory of the spherical pendulum for subsequent extraction from its observable experimental curve.

In the experiments10)-13) pendulums were used in the complete ellipsoidal mode, and a system for automatic restarting of pendulum oscillations was employed. However, a comparison with the theoretically predicted trajectory was not completed. In addition there was a large number of errors. On the contrary, the authors of the proposal13) presented a long list of dangerous sources of experimental errors and discussed methods to circumvent them. But this consideration has been limited by trajectories with small ellipticity (b/a) \ll 1. They found anisotropies of different type as more dangerous hindrances, including anisotropy of frequencies, friction and stochastic vibrations.

Pippard16) continued a discussion of the feasibility of the experiment and pro-
posed a parametrical pumping for both goals to compensate friction losses (in Ref 13) a high $Q$ pendulum with one year relaxation was supposed) and to maintain a small ellipticity.

We should remark here that authors of papers$^{13,16}$ did not appeal to any rigorous theory of the spherical pendulum and made conclusions based on half-phenomenological considerations.

Among serious mathematical works we can list the paper of Miles$^8$ (and references therein) where nonlinear Hamilton equations of the spherical pendulum with parametric pumping were solved numerically. But Miles concentrated on a particular problem of the pendulum's chaotic regimes, searching for bifurcation points etc., i.e., he considered regimes with large amplitudes which are unlikely to be applicable for measurements of fine fundamental gravity phenomena.

After this brief review of the problem of the spherical pendulum, we can formulate the objectives of our paper as follows. The principal goal is to develop an analytical description of the nonlinear device of our interest using as much as possible asymptotical methods of the theory of nonlinear oscillations. It would be desirable to obtain a complete analytical trajectory description, but the particular aim is a derivation of a general formula for the velocity of precession, depending on arbitrary external forces.

In addition to the analysis in Refs. 10) and 16), we are considering the very important effect of "sympathetic oscillations" of the support, which together with the elastic anisotropy could produce nontrivial precession features.

We believe that our results allow one to give a plausible interpretation of the anomalies observed in the experiment$^{10}$, as well as to formulate additional requirements for a set up to measure a gravimagnetic effect of general relativity.

§2. Equations of motion

The equation of motion for the mathematical spherical pendulum might be derived in different ways.$^1$-$^3$ Here we prefer to use the Lagrangian method of "independent coefficients", so that according to (Fig.1) the equation of motion is

$$m\ddot{r} = mg + A\text{Grad}F + f = 0,$$

$$F(x, y, z) = x^2 + y^2 + z^2 - l^2 = 0, \quad (1)$$

where $A$ is the Lagrange parameter, $F(x, y, z)$ the equation of the boundary surface, $g$ the gravity acceleration and $f$ some perturbation forces.

The four equations (1) constitute a complete system for the pendulum coordinates and the parameter $A$ so that, using the approximation $z \approx l[1 - (x^2 + y^2)/(2l^2)]$ and excluding $A$, one can easily come to the "plane equations" of the spherical pendulum in terms of non-dimensional variables $\xi = x/l$, $\eta = y/l$, $\tau = \omega t$ with accuracy.
of order $\xi^2, \eta^2$:

\[
\begin{align*}
\xi'' + \omega^2 \xi &= -2\Delta \xi - (1/2) N \xi + f_\xi, \\
\eta'' + \omega^2 \eta &= -2\Delta \eta - (1/2) N \eta + f_\eta.
\end{align*}
\tag{2}
\]

In (2) the two new definitions $N$ and $\Delta$ were introduced: $N = (\xi^2 + \eta^2) + (\xi^2 + \eta^2)^\prime$ the so-called "boundary rigidity", $\Delta = (\omega - \omega_0)/\omega$ a "detuning" of the current frequency $\omega$ from the linear pendulum resonance frequency $\omega_0 = (g/l)^{1/2}$. We suppose that $\omega \sim \omega_0$, or $|\Delta| \ll 1$.

For the linear isotropic "Foucault pendulum" one must set $N = 0$, $\Delta = 0$ and to take components of the Coriolis force as perturbations $f_i(i = \xi, \eta)$:

\[
f_i = -2 \{\varepsilon \eta', -\varepsilon \xi'\},
\tag{3}
\]

where the kinematic parameter $\varepsilon$ depends on the Earth spin angular velocity $\omega_E$ and $\lambda$ — the geographical latitude of the laboratory.

To analyze the equations (2) in a more general case, it is convenient to use a quasi-harmonic approximation of the "plane variables", introducing "slow" amplitudes $A_i$ and phases $\theta_i$ so that $|A'/A| \sim |\theta'/\theta| \ll 1$ and

\[
\xi = A_\xi \cos (\tau + \theta_\xi), \quad \eta = A_\eta \cos (\tau + \theta_\eta).
\tag{4}
\]

Then, after substituting (4) into (2) and performing the averaging over the "fast" period $2\pi$ in a standard way,$^{17}$ one comes to the following so-called "shortened equations" for the slow variables:

\[
\begin{align*}
A_\xi' &= (3/8)\Phi A_\eta \cos \theta_\eta - M_t \{f_\xi \sin(\tau + \theta_\xi)\}, \\
A_\xi \theta_\eta' &= \left(\Delta - \frac{E}{16}\right) A_\xi - (3/8)\Phi A_\eta \sin \theta_\xi - M_t \{f_\xi \cos(\tau + \theta_\xi)\}, \\
A_\eta' &= -(3/8)\Phi A_\xi \cos \theta_\xi - M_t \{f_\eta \sin(\tau + \theta_\eta)\}, \\
A_\eta \theta_\xi' &= \left(\Delta - \frac{E}{16}\right) A_\eta - (3/8)\Phi A_\xi \sin \theta_\eta - M_t \{f_\eta \cos(\tau + \theta_\eta)\}.
\end{align*}
\tag{5}
\]

Here we have the additional important definitions:

\[
\begin{align*}
E &= A_\xi^2 + A_\eta^2 \quad — \text{the energy of the pendulum}, \\
\Phi &= A_\xi A_\eta \sin \theta \quad — \text{the momentum of the motion}, \\
\theta &= \theta_\eta - \theta_\xi \quad — \text{the phase difference}, \\
M_t \{\ldots\} \quad — \text{the time averaging operator}.
\end{align*}
\tag{6}
\]

It is worth remarking that the functions $E$ and $\Phi$ are integrals of motion for the equation (6), i.e. $E' = 0$, $\Phi' = 0$. For simplicity, below we suppose a special "detuning" $\Delta = E/16$.

In the quasi-harmonical approximation (4) a pendulum trajectory on the horizontal plane presents a slow rotated ellipse, so that the ellipse equation is

\[
(\xi^2/A_\xi^2) - 2(\xi \eta/A_\xi A_\eta) \cos \theta + (\eta^2/A_\eta^2) - \sin^2 \theta = 0.
\tag{7}
\]
The angle $\varphi$ between the positive direction of the $OX$ axis and the major axis of the ellipse is defined as

$$
\varphi = \int \Omega(\tau) d\tau = \frac{1}{2} \arctan \frac{2 A_\xi A_\eta \cos \theta}{A_\xi^2 - A_\eta^2}.
$$

(8)

To determine the angular velocity $\Omega$ of the major axis rotation in the horizontal plane $(x, y)$ (or the so-called "rotation velocity of the swing plane" for the ellipse with large oblateness) it is necessary to differentiate both parts of (8), after which we obtain

$$
\Omega = \varphi' = \frac{E \cos \theta (A_\xi A_\eta' - A_\eta A_\xi') + \Phi \theta' (A_\eta^2 - A_\xi^2)}{E^2 - 4 \Phi^2}.
$$

(9)

This equation together with the system of shortened equations (5) presents a solution of the problem of the "precession velocity" in a more general form.

Below, Eqs. (9) and (5) are used to analyse some perturbations in the rotation of the swing plane under the action of different perturbations, such as synchronous support bendings, so-called "sympathetic oscillations", suspension anisotropy, viscous friction, vibration of the suspension, and other forces.

§3. Sympathetic oscillations of the anisotropic support

The sympathetic oscillations of support are displacements of the pendulum suspension point relative to the direction of pendulum deflections. For the spherical pendulum, having two degrees of freedom, elastic characteristics toward the axes $X$ and $Y$ generally could be different. In particular, an anisotropy of the suspension for Allaise’s pendulum can be produced by specific features of a "ball support".

Suppose that displacements of the support point are proportional to those of the pendulum body:

$$
\vec{S} = \{\alpha_\xi \xi, \alpha_\eta \eta\},
$$

(10)

where $\alpha_\xi, \alpha_\eta < 1$ are coupling coefficients depending on elastic properties of the support. Generally $\alpha_\xi \neq \alpha_\eta$.

This can be seen in the equations of motion (2) and (5) as some equivalent inertial force $\vec{f} = \vec{S}'' \approx -|\vec{S}|$.

Thus taking into consideration expression (3), one has for the total force:

$$
f_i = -2 \left\{ \left( \varepsilon \eta' + \frac{1}{2} \alpha_\xi \xi'' \right), \left( -\varepsilon \xi' + \frac{1}{2} \alpha_\eta \eta'' \right) \right\}.
$$

(11)

Then from the shortened equations (5) with (6) and (11) it is possible to find (if $\theta = \theta_\eta - \theta_\xi, \ \alpha = \alpha_\eta - \alpha_\xi$)

$$
\theta' = -\left( (3/8) \Phi - \varepsilon \right) \Phi^{-1} (A_\eta^2 - A_\xi^2) \sin^2 \theta + \alpha.
$$

(12)

Substitution of (12) into (9), referring to (5) results in

$$
\Omega = \varepsilon - \frac{3}{8} \Phi + \frac{\Phi \psi}{E^2 - 4 \Phi^2}
$$

(13)
where $\Psi = \alpha (A^2_\xi - A^2_\eta)$.

It is useful to remark that the effect of sympathetic oscillations of the support according to (11) can be considered as a particular case of the "elastic anisotropy" of the spherical pendulum analyzed in Refs. 13 and 16); formula (13) remains valid for any $|\alpha_i| \ll 1$.

Let us show that the functions $\Phi$ and $\Psi$ are not independent, but one can be expressed in terms of each other. Indeed, after differentiation one has

$$
\Phi' = \alpha A_\xi A_\eta \cos \theta,
$$

$$
\Psi' = -4 ((3/8)\Phi - \varepsilon) \alpha A_\xi A_\eta \cos \theta,
$$

$$
\Psi = -4 ((3/16)\Phi^2 - \varepsilon \Phi) + C_0,
$$

$$
C_0 = \alpha (A^2_\xi - A^2_\eta) \alpha + 4 ((3/16)\Phi^2 - \varepsilon \Phi)_0,
$$

where the zero index marks values referred to the initial starting time of the pendulum.

For a conservative pendulum with $E = \text{const}$, $\varepsilon = \text{const}$, in order to calculate the angular velocity of precession $\Omega$ in (13) it is sufficient to determine the function $\Phi(\tau)$ in a clear form. To do this let us differentiate $\Phi(\tau)$, taking into account (12) and (14):

$$
\Phi'' + \alpha^2 \Phi = \alpha (A^2_\xi - A^2_\eta) ((3/8)\Phi - \varepsilon) = ((3/8)\Phi - \varepsilon) \Psi.
$$

Let us then choose initial conditions for Eq. (17) in a special way as

$$
\Phi(0) = \Phi_0 = \max \Phi(\tau) = (A_\xi A_\eta \sin \theta)_0, \quad \Phi'(0) = \alpha (A_\xi A_\eta \cos \theta) = 0.
$$

Equation (17) with (15), (16) and (18) is a nonlinear equation with separated variables (see the Appendix).

The simplest solution corresponds to the case in which the pendulum is placed in the equatorial area with $\varepsilon = (\omega_E/\omega) \sin \lambda \to 0$. (Here we wish to note in the experiments$^{11,12}$ performed in Mexico and Brasil, $\varepsilon \approx 0$.) For $\varepsilon = 0$ a solution of Eq. (17) might be presented in the form

$$
\Phi(\tau) = \Phi_0 \cn(\sigma \tau, k),
$$

where $\cn(u, k)$ is the elliptic cosine with parameters

$$
\sigma^2 = 2 \left( \frac{3}{8} \right)^2 \Phi^2_0 + \left( \alpha^2 - \frac{3}{8} C_0 \right), \quad k^2 = \left( \frac{3}{8} \right)^2 \left( \frac{\Phi^2_0}{\sigma^2} \right).
$$

The final expression for the precession angular velocity is

$$
\Omega = -\Phi_0 \cn(\sigma \tau, k) \left[ \frac{3}{8} \frac{C_0 - (3/4)\Phi^2_0 \cn^2(\sigma \tau, k)}{E^2 - 4\Phi^2_0 \cn^2(\sigma \tau, k)} \right].
$$
The dependence of $\Omega$ on the anisotropy parameter $\alpha$ and initial conditions turns out to be complex; $\Omega$ can change from a nearly harmonic form up to the complex periodic process presented for the example in Fig. 2. The period $T$ of the rotation movement of the "swing plane" is defined by the full elliptical integral of first kind: $T = (4/\sigma)K(k)$ (the Appendix).

For an isotropic suspension, when $\alpha = \alpha_1 - \alpha_2 = 0$ and $\Phi = \Phi_0 = \text{const}$, one finds:

$$\Omega = -(3/8)\Phi_0 = -(3/8)(A_\xi A_\eta \sin \theta)_0. \quad (22)$$

This formula was obtained by Krylov$^3$ and also by Olsson.$^6$ Thus, for an isotropic spherical pendulum, the sympathetic oscillations produce only the typical nonlinear correction of the precession velocity depending on initial conditions.

A behavior of the spherical pendulum in the general case $\varepsilon \neq 0$ is strongly unharmonical. The corresponding analytical considerations prove to be very complex, and the logical scheme for this analysis is presented in the Appendix.

§4. Influence of stochastic forces

Let us consider a general case with two independent external forces along the coordinate axes $f_\xi$ and $f_\eta$ which can be functions of the dimensionless time $\tau$. Then formula (9) for the precession angular velocity can be reduced to another form (taking into account Eq. (5)),

$$\Omega = \varepsilon - \frac{3}{8} \Phi + \frac{A_\eta < f_\xi \sin(\gamma_\xi + \beta) > - A_\xi < f_\eta \sin(\gamma_\eta + \beta) >}{\sqrt{E^2 - 4\Phi^2}}, \quad (23)$$

where $\gamma_i = \tau_i + \theta_i$, $i = \xi, \eta$, and $\beta$ is an additional phase shift, so that

$$\cos \beta = \frac{E \cos \theta}{\sqrt{E^2 - 4\Phi^2}}, \quad \sin \beta = \frac{(A_\eta^2 - A_\xi^2) \sin \theta}{\sqrt{E^2 - 4\Phi^2}}. \quad (24)$$

Then, having in mind that $\Phi = A_\xi A_\eta \sin \theta$, it is possible to find a new expression for $\Phi'$:

$$\Phi' = (A_\xi A_\eta \sin \theta)' = A_\eta < f_\xi \cos \gamma_\eta > - A_\xi < f_\eta \cos \gamma_\xi >. \quad (25)$$

It is worth noting that Eq. (14) is a particular case of (25) under the special choice of external forces (11).

a) When the source of the external forces is a viscous friction with a friction coefficient $\Gamma$ one may write

![Fig. 2. Angular velocity of pendulum with anisotropic suspension as a function of time.](image)
The viscous friction has an effect only on the evolution of the function $\Phi$. From (25) with (26) and (5) it is easy to find $\Phi = \Phi_0 e^{-2 \Gamma \tau}$. Thus the dissipation forces $f_i(\tau)$ lead to an additional variation of the velocity $\Omega$ compared to unperturbed case $\Gamma = 0$. From a physical point of view the reason for this is clear. The dissipation depresses amplitudes $A_\xi$ and $A_\eta$ so that the function $\Phi = A_\xi A_\eta \sin \theta$ also decreases exponentially.

b) Let $f_\xi$ and $f_\eta$ be independent stationary stochastic processes of the "white noise" type with correlation functions

$$B_i(\tau_2 - \tau_1) = M\{f_i(\tau_1), f_i(\tau_2)\} = N_i \delta(\tau_2 - \tau_1),$$

where $i = \xi, \eta$, $M\{\ldots\}$ is the averaging statistical operator and generally $N_\xi \neq N_\eta$.

In this case the "shortened equation" (5) should be reduced to the form

$$A'_\xi = \left(\frac{3}{8} \Phi - \varepsilon\right) A_\eta \cos \theta - \frac{N_\xi}{4 A_\xi} + n_{a\xi},$$

$$\theta'_\xi = \left(\Delta - \frac{E}{16}\right) \left(\frac{3}{8} \Phi - \varepsilon\right) \frac{A_\eta}{A_\xi} \sin \theta - \frac{N_\xi}{4 A_\xi} + n_{a\xi},$$

$$A'_\eta = -\left(\frac{3}{8} \Phi - \varepsilon\right) A_\xi \cos \theta - \frac{N_\eta}{4 A_\eta} + n_{a\eta},$$

$$\theta'_\eta = \left(\Delta - \frac{E}{16}\right) \left(\frac{3}{8} \Phi - \varepsilon\right) \frac{A_\eta}{A_\xi} \sin \theta - \frac{N_\xi}{4 A_\xi} + n_{a\xi}.$$

Here $n_{a\xi}, n_{a\eta}, n_{a\xi}, n_{a\eta}$ are independent $\delta$-correlated noise functions:

$$M\{n_{a\xi}(\tau_1) n_{a\xi}(\tau_2)\} = M\{n_{a\xi}(\tau_1) n_{a\xi}(\tau_2)\} = 2 N_\xi \delta(\tau_2 - \tau_1),$$

$$M\{n_{a\eta}(\tau_1) n_{a\eta}(\tau_2)\} = M\{n_{a\eta}(\tau_1) n_{a\eta}(\tau_2)\} = 2 N_\eta \delta(\tau_2 - \tau_1).$$

It is difficult to obtain a solution of the system (29) in the general case, because of its nonlinear difficult stochastic equations. But we can analyze the behavior of this system during its initial evolution performing a linearization of the Eq. (29) around the initial conditions: $A_\xi(0) = A_0; A_\eta(0) = A_0$. Using such "a linearized solution" it is easy to find from (23) and (25) an expression for the perturbed $\Omega$:

$$\Omega \approx \varepsilon + \frac{A_0}{2} \left[\frac{3}{8} \int_0^\tau f_{\eta c} d\tau + \frac{1}{A_0^2} f_{\eta s}\right],$$

$$f_{\eta c} = 2 < f_\eta \cos(\tau + \theta_0)>, \quad f_{\eta s} = -2 < f_\eta \sin(\tau + \theta_0) >.$$
The first term in (31) represents the well-known "Foucault precession". The second term describes a diffusion perturbation of the velocity of the "swing plane" due to slow variations of the function $\Phi$. The third term represents a "white noise" of $\Omega$.

Thus we have nonstationary variations of the angular velocity $\Omega(\tau)$ as some stochastic $\delta$-correlated process with the correlation function

$$ M\{\Omega(\tau_1)\Omega(\tau_2)\} = \mathcal{D}\delta(\tau), $$

$$ \mathcal{D} = (1/2A_0^2)N_\eta. $$

Expression (32) leads to the estimation

$$ \sigma_\Omega \approx (\mathcal{D}/\tau_m)^{1/2}, $$

of the measurement accuracy for the angular velocity $\Omega(\tau)$, where $\tau_m$ is a measurement time.

c) Let us make a remark concerning the case of dissipative losses and dynamic instability of the pendulum.

Miles investigated in detail the principal possibility of dynamic chaostisation for a spherical pendulum with viscous friction in the presence of induced harmonic oscillations of the pendulum suspension point along the $X$ axis:

$$ \tilde{f}_{s\xi} = -\delta_\xi = \omega_f^2 f_0 \cos \omega_f \tau, \quad \omega_f \approx \omega. $$

The Poincaré point method was used to investigate stochastic behavior of the pendulum with an anisotropic suspension $\alpha \neq 0$ and viscous friction. Under the action of an external force, it is suitable to renormalize the equations of system (2) to an external force frequency $\omega_f$ making the transition to another dimensionless time $\tau_m = (1/2)x_0^{2/3} \tau$, as well as to express $\xi$ and $\eta$ through its quadrature components $a_\xi, a_\eta, b_\xi$ and $b_\eta$ according to the formulas

$$ \xi = x_0^{-1/3}(a_\xi \cos \omega_f \tau_m + b_\xi \sin \omega_f \tau_m), \quad \eta = x_0^{-1/3}(a_\eta \cos \omega_f \tau_m + b_\eta \sin \omega_f \tau_m), $$

with the equations for quadrature components

$$ a'_\xi = -\delta_m a_\xi - \left(\nu + \frac{1}{8} E\right) b_\xi - \frac{3}{4} Ma_\xi + \alpha_{m\xi} b_\xi, $$

$$ b'_\xi = -\delta_m b_\xi - \left(\nu + \frac{1}{8} E\right) a_\xi - \frac{3}{4} Mb_\xi + \alpha_{m\xi} a_\xi, $$

$$ a'_\eta = -\delta_m a_\eta - \left(\nu + \frac{1}{8} E\right) b_\eta - \frac{3}{4} Ma_\eta + \alpha_{m\xi} a_\xi, $$

$$ b'_\eta = -\delta_m b_\eta - \left(\nu + \frac{1}{8} E\right) a_\eta - \frac{3}{4} Mb_\xi + \alpha_{m\xi} a_\eta, $$

where $\nu = x_0^{-2/3}(\Omega^2 - \omega^2)/\Omega^2$, $\delta_m = 2\delta x_0^{-2/3}(\omega/\Omega)$, $x_0$ is the amplitude of oscillations, and also $E = a_\xi^2 + a_\eta^2 + b_\xi^2 + b_\eta^2$ and $M = a_\eta b_\xi - a_\xi b_\eta$.

The system (36) is similar to that explored by Miles, but in addition it also takes into account the anisotropy of the suspension. For computer simulation of Eq. (36)
the fourth-order Runge-Kutta method was used. For small anisotropies \((\alpha_m \leq 10^{-4})\), the phase portrait of the system has not been practically distinguished from that obtained by Miles without anisotropy. However, increasing the anisotropy produces an essential effect upon the dynamics of the pendulum. Both typical nonlinear modes in the pendulum behavior, bifurcations of period of different orders and transitions to dynamic chaos were observed as a function of \(\alpha_m\). The results of such numerical experiments show that for the initial conditions, typical for experiments\(^1\),\(^2\) the increase of anisotropies produces a significant increase in the duration of the transmission process, during which the pendulum fluctuations are accompanied by jumps of the rotation angle.

§5. Discussion and conclusions

Our analysis of spherical pendulum behavior was carried out on the basis of averaging methods, the so-called van der Pol approximation\(^17\) when only a principal harmonic component of nonlinear system was taken into account. Such an approach allows one to find the “nonlinear correction” to the angular velocity \(\Omega\) of an “swing plane” of the pendulum.

In this paper we concentrated on the influence of the suspension anisotropy, dissipative losses and external stochastic forces on the pendulum dynamics. For the case of dissipation and anisotropy it is possible to derive the analytic formula for \(\Omega\) as a function of anisotropy parameters and friction coefficients \((21)\) and \((27)\). However, the system dynamics became more complex under the action of wideband inertial disturbances of the “white noise” type. In this case the averaging procedure has to be performed in two steps: first, the “time averaging” of terms containing no direct noise variables and second, the “statistical averaging” of stochastic sources, taking into account then possible correlation with phase shifts \(\Delta \theta_x\) and \(\Delta \theta_y\). Equations resulting from this procedure permit one generally to utilize the apparatus of Markov’s processes based on the Fokker-Plank equation.\(^19\) But an analytical attempt to do this meets technical problems. For this reason in our analysis we were limited by a “linearized theory” with a region of validity around the initial conditions. A “new element” of this consideration resulted in an additional drift (as compared to the diffusional drift) of the precession velocity \(\Omega\).

A more detailed analysis of spherical pendulum dynamics leads us to the following conclusions:

1. The angular precession velocity \(\Omega\) and trajectory of the ideal spherical pendulum in the absence of perturbations depends on the initial conditions only. Small variations of these conditions result in different values of the precession velocity. This was observed in experiments\(^11\),\(^12\) where the “restarting methodics” were utilized.

2. The sympathetic oscillations of the support (typical for all gravimetric instruments) under the anisotropy of the suspension complicate the pendulum motion. A finite value of the anisotropy parameter \(\alpha\) could result in a nonmonotonous acceleration of the precession velocity. This may be incorrectly interpreted as a “result of gravitational influence”.\(^10\) The period of modulation depends on the anisotropy of elastic parameters of suspension and initial conditions and can be presented by the
elliptical integral of the first kind, $K(\varphi, k)$. In the general case this dependence is described by the non-linear differential equation. The precession velocity becomes a very complex periodic function so that even the direction of the precession can change its sign sharply (the so-called “anti-Foucault” precession). This effect was also observed in experiments.

3. The same joint influence of support anisotropies and the Foucault effect imitates completely the Lense-Thirring effect, which appears as some anomaly precession of the pendulum swing plane with an angular velocity

$$\Omega_{LT} = \omega_E (1 + r_g / R),$$

where $R$ and $r_g$ are the physical and gravitational radii of Earth, respectively. This means that for performing an experiment of type Ref. 13, strong restrictions on the initial conditions and parameters of the support must be provided:

$$|\alpha (A^2_\eta - A^2_\xi)_0 \Phi_0| \approx A^2_\xi |\alpha \Phi_0| \ll 10^{-10}.$$

4. Dissipation decreases the role of nonlinear effects. The angular precession velocity relaxes exponentially with a characteristic time defined by frictional coefficients (26).

5. The influence of wideband stochastic forces in the linear approximation leads to the additional variation of the precession angle $\varphi = \int_0^T \Omega(\tau) d\tau$ with the diffusion coefficient $D$ described by the formula (32). Also an additional drift of $\Omega(\tau)$ appears; this is a nonstationary stochastic process depending on initial conditions.

6. For finite amplitudes of perturbations of support point (produced for example by seismic noise) dynamical chaostisation of the pendulum may appear. This also limits possibilities of application of such instruments in gravitational experiments. Numerical integrating was performed to investigate bifurcation phenomena for pendulums with anisotropic suspensions and regular vibrations of the suspension point. This allows us to generalize the known results, received by Miles. The anisotropy of the support produces an appreciable influence upon the moments of transition of the pendulum to dynamical chaos.

As a final remark it is interesting to consider another application of our results. For the classical Foucault pendulum with a rigid arm the suspension point is supposed to represent a rotational degree of freedom. Such a mechanism has to be specially provided in the experimental set-ups. But any ordinary pendulum on a soft single filament will display a “Foucault type behavior”, because it is difficult to induce oscillations in a single plane. Oscillations must be presented at least in two planes if the filament has a corresponding anisotropic bending modulus. Thus a pendulum trajectory could be elliptical with the complex features described above. This effect produces some problem for measurement of a high $Q$-factor of a mirror’s suspensions in gravitational wave free mass interferometers, and a special technique has been developed to avoid its influence within the accuracy of measurements. We believe that the calculations carried out in the present paper can also be useful for analysis of these experiments.
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Appendix

Equation (17) can be transformed into another form using Eqs. (14), (16), namely

\[ \phi'' + a_1 \phi + a_2 \phi^2 + a_3 \phi^3 + a_0 = 0, \tag{A.1} \]

where for simplification the coefficients \( a_i \) are introduced so that

\[ a_1 = (\alpha^2 + 4\varepsilon^2 - (3/8)C_0), \quad a_2 = - (9/4)\varepsilon, \quad a_3 = 2(3/8)^2, \quad a_0 = C_0 \varepsilon. \tag{A.2} \]

When \( a_3 > 0 \), Duffing’s equation (A.1) must have a periodic solution. It is well known that Eq. (A.1) can be factorized. After this factorization, variables are separated. Following standard methods, one introduces \( v = d\phi/d\tau = \phi' \). Then (A.1) reduces to

\[ dv^2 = -2 (a_1 \phi + a_2 \phi^2 + a_3 \phi^3 + a_0) d\phi. \]

Hence

\[ v^2 = -2 \left( \frac{a_1}{2} \phi^2 + \frac{a_2}{3} \phi^3 + \frac{a_3}{4} \phi^4 + a_0 \phi \right) + C_v. \tag{A.3} \]

The integration constant \( C_v \) can be defined from the conditions

\[ \Phi_0 = \max_\tau \Phi(\tau) \quad \text{and} \quad \Phi' = v = 0. \]

If \( \Phi = \Phi_0 \) then

\[ C_v = 2 \left( \frac{a_1}{2} \phi_0^2 + \frac{a_2}{3} \phi_0^3 + \frac{a_3}{4} \phi_0^4 + a_0 \phi_0 \right). \tag{A.4} \]

From (A.3) and (A.4) one has

\[ v^2 = \left( \frac{d\phi}{d\tau} \right)^2 = \frac{a_3}{2} (\Phi_0 - \Phi) Q(\Phi), \tag{A.5a} \]

\[ Q(\Phi) = \phi^3 + a\phi^2 + b\phi + C. \tag{A.5b} \]

Thus the \( v \)-variable is defined by the formula

\[ v = \frac{d\phi}{d\tau} = \pm \sqrt{\frac{a_3}{2} (\Phi_0 - \Phi) \prod (\Phi - \Phi_n)}, \tag{A.6} \]
where \( \Phi_n \) are roots of the cubic equation \( Q(\Phi) = 0 \). A solution of this cubic equation for the case of real coefficients \((a, b, c)\) is given by Cardano's formulas.\(^{18}\) Taking into account (A·5a) one has

\[
\tau = \pm \int \frac{d\Phi}{\sqrt{(a_3/2)(\Phi - \Phi_n) \prod (\Phi - \Phi_m)}} + C\tau. \tag{A·7}
\]

Supposing that \( \Phi(\tau = 0) = \Phi_0 \) we can omit the integration constant \( C\tau = 0 \), and choose the sign "+" for \( \tau \geq 0 \).

The integrand (A·7) in the case of definite roots \( \Phi_n \) can be found in mathematical reference books (for example, Ref. 18)). This turns out to be the elliptical integral \( K(\varphi, k)\)\(^{18}\) in a general case. But simplest results correspond to the case of a spherical pendulum located in the equatorial zone where \( \varepsilon = (\omega_2/\omega) \sin \lambda \to 0 \). Then \( a_2 = a_0 = 0 \), and the solution of the equation \( Q(\Phi) = 0 \) can be easily obtained. We thus have the result

\[
\tau = \int_{\Phi_0}^{\Phi} \frac{d\Phi}{\sqrt{(a_3/2)(\Phi^2 - \Phi_0^2)(\Phi^2 - \Phi_m^2 + 2(a_1/a_3))}}. \tag{A·8}
\]

For the standard change of variables \( \Phi = \Phi_0 \cos \varphi \), this is reduced to

\[
\tau = \left( \frac{1}{\sqrt{\sigma^2}} K(\varphi, k) \right), \quad \sigma^2 = a_2 \Phi_0^2 + a_1, \quad k = \frac{a_2 \Phi_0^2}{2\sigma^2}, \tag{A·9}
\]

from which \( \Phi = \Phi_0 \cos \varphi = \Phi_0 \text{cn}(\sigma\tau, k) \) is the elliptic cosine.\(^{18}\)

References

5) W. B. Somerville, Astron. Soc. 13 (1972), 40.