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Galilei Covariance and (4,1)-de Sitter Space

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A vector space $\mathcal{G}$ is introduced such that the Galilei transformations are considered linear mappings in this manifold. The covariant structure of the Galilei Group (Y. Takahashi, Fortschr. Phys. **36** (1988), 63, 83) is derived and the tensor analysis is developed. It is shown that the Euclidean space is embedded in the (4,1)-de Sitter space through $\mathcal{G}$. This is an interesting and useful property, in particular, for the analysis carried out for the Lie algebra of the generators of linear transformations in $\mathcal{G}$.

The context of Galilean symmetries constitute a natural setting for formulations of non-relativistic theories, with particular signficance to the study of the condensed matter physics.[1)-6)] Unlike the Poincaré group, however, the representations of the Galilei group (G) have not been sufficiently developed,[5)] even though there is a wealth of information about non-Lorentzian physics that could benefit greately from such studies.

One characteristic making such a development difficult is the intricate structure of G, characterized by eleven parameters: three spatial rotations, three spatial translations, three boosts, one time translation, and one central extension (necessary to find physical representations). The natural representation, nevertheless, is described by 10 parameters and specified by

$$\bar{\boldsymbol{x}} = R\boldsymbol{x} + \boldsymbol{v}t + \boldsymbol{a}, \tag{1}$$
$$\bar{t} = t + b. \tag{2}$$

Usually G is introduced without the metrical vector space in which the transformations given by Eqs. (1) and (2) are defined, as is the case for the group $O(3)$, defined on the $\mathcal{R}^3$-space or the Poincaré group, defined as linear transformations in the Minkowski space. This lack of a Galilean metric-vector space has a consequence that a ray representation of G is not trivially reducible to a faithful representation. Therefore it is interesting, for the study of the Galilei symmetries, to specify the manifold underlying the Galilean transformations. The objective of this paper is to give the Galilean group also as a linear isometry group, similar to the case, for instance, of the Poincaré group.

The elucidation of the tensor structure of the Galilean transformations was undertaken some time ago by one of the present authors.[7),8)] Such a structure is based in a five-dimensional formulation of the Galilei transformations, and has been motived by the development of Galilei invariant field theory. For instance, this formalism has been used to introduce generalized Schrödinger equations and to derive a non-linear Galilei invariant field equation, from which the rearrangement of symmetries describing rotons and phonons has been studied.

Here, developments of that covariant approach to the Galilei group are presented. In particular, we introduce the tensor formulation, stressing its manifold characteristics in order to develop the manifold analysis. In this sense, we construct representations of the Galilei Lie algebra on such a manifold (say $\mathcal{G}$), taking advantage of the (if not intriguing, of least practical) fact that $\mathcal{G}$ can be considered as an embedding of the 3-dimensional Euclidean space ($\mathcal{E}$) in a $(4,1)$ de Sitter space. [9]

Considering kinematic groups, the Galilei group has been studied previously via a Wigner-Inönöu contraction of the Poincaré group, which is, in turn, contracted from the de Sitter group. [10] This, however, is not the case for the approach developed here, where the concept of embedding, involving geometrical structures without any limiting process, is used. This allows us to establish a direct link between the Galilei and de Sitter groups.

Let us begin by observing that in $\mathcal{E}$, the metric space defined on $\mathcal{R}^3$, the distance between two points is preserved under linear transformations. That is, given two vectors $\boldsymbol{x} = (x^1, x^2, x^3)$ and $\boldsymbol{y} = (y^1, y^2, y^3)$, where $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{E}$, then $r^2 = \boldsymbol{x}^2 + \boldsymbol{y}^2 - 2\boldsymbol{x}\cdot\boldsymbol{y}$ is invariant under translations and rotations. In a physical system described by Galilei symmetries, two types of translations of G occur in $\mathcal{E}$. However, one of these, the boost, is defined via an external parameter, the time t (see Eqs. (1) and (2)); this is a central aspect of G.

We can consider, therefore, using the form of the distance $\boldsymbol{r}$ in the Euclidean space to embed $\mathcal{E}$ in a lager manifold, say $\mathcal{G}$, such that Eqs. (1) and (2) can be considered as linear transformations in $\mathcal{G}$. This can be achieved, indeed, if we observe that

$$s^2 = -\frac{1}{2}r^2 = -t\frac{\boldsymbol{x}^2}{2t} - t\frac{\boldsymbol{y}^2}{2t} + \boldsymbol{x}\cdot\boldsymbol{y} \tag{3}$$

is no more than the inner product of two particular vectors of a space $\mathcal{G}$, which is defined as follows. Let $\mathcal{G}$ be a 5-dimensional metric space with an arbitrary vector denoted by $x = (x^1, x^2, x^3, x^4, x^5) = (\boldsymbol{x}, x^4, x^5)$. The inner product in $\mathcal{G}$ is defined by

$$\begin{aligned}(x|y) &= \eta_{\mu\nu}x^\mu y^\nu \\ &= \sum_{i=1}^{3} x^i y^i - x^4 y^5 - x^5 y^4,\end{aligned} \tag{4}$$

where $x, y \in \mathcal{G}$ and $\eta_{\mu\nu}$, the metric, is given by

$$(\eta_{\mu\nu}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}. \tag{5}$$

(Latin indices represent components of vectors in $\mathcal{E}$, as in Eq. (4).) [7),8),11)]

Note that s^2, defined in Eq. (3), is a particular case of the inner product in $\mathcal{G}$. That this is the case can be seen by writing

$$x^5 = \frac{\boldsymbol{x}^2}{2t}, \quad y^5 = \frac{\boldsymbol{y}^2}{2t} \quad \text{and} \quad x^4 = y^4 = t, \tag{6}$$

in Eq. (4). (In order to adjust the physical units of space and time, we can define, for instance, $x^4 = vt$, with $v = 1m/s$.)

Let $\{e_\mu\} = \{e_1, \cdots, e_5\}$ be a basis vector of $\mathcal{G}$, such that $x = x^\mu e_\mu$ and $y = y^\mu e_\mu$. Then, from Eq. (4) it follows that $(e_\mu|e_\nu) = \eta_{\mu\nu} = \eta_{\nu\mu}$. In addition, the dual structure of $\mathcal{G}$ can be introduced. Consider $\mathcal{G}^* = \{w_1, w_2, \cdots\}$ the set of linear forms on $\mathcal{G}$; that is, $w : \mathcal{G} \mapsto \mathcal{R}$, so that $(w_1 + aw_2)\, x = w_1(x) + aw_2(x),\ a \in \mathcal{R}$, and

$$w(x) = x^\mu w(e_\mu) = w_\mu x^\mu, \tag{7}$$

where $w_\mu = w(e_\mu)$. The space $\mathcal{G}^*$ is defined by the following set of 1-forms. Therefore, from Eq. (7), we can write $w = w_\mu e^\mu$, so that $\{e^\mu\}$ is a dual basis of $\mathcal{G}$.

The metric $\eta_{\mu\nu}$ can be used to properly define the operation of raising and lowering of indices. In order to do so, first one uses Eq. (4) to introduce a natural 1-form (also called the 'natural pairing'[12)]) defined by $x^*(y) \equiv (x|y)$. Second, writing $x = x^\mu e_\mu$ and $x^* = x_\nu e^\nu$, it follows that $x^*(y) = x_\mu y^\nu e^\mu(e_\nu)$. This leads to $e^\mu(e_\nu) = \delta^\mu{}_\nu$. Considering then the definition of x^* and the fact that y is an arbitrary vector, the operation of lowering indices is established; that is $x_\mu = \eta_{\mu\nu} x^\nu$. Introducing $(\eta^{\mu\nu})^{-1} = (\eta_{\mu\nu})$, the operation of raising indices is defind according to $x^\mu = \eta^{\mu\nu} x_\nu$. As a result, Eq. (4) can be written as $(x|y) = x^\mu y_\mu = x_\mu y^\mu$, such that

$$\begin{aligned} e^i(x) &= x^i = x_i, \quad i = 1,2,3, \\ e^4(x) &= x^4 = -x_5, \\ e^5(x) &= x^5 = -x_4. \end{aligned}$$

The norm of a vector in $\mathcal{G}$ is defined as $\|x\| = (\boldsymbol{x})^2 + x_4 x^4 + x_5 x^5 = (\boldsymbol{x})^2 - 2x^4 x^5$. If $\|x\| > 0$ and x^4 and x^5 are real numbers with the same sign, then $\boldsymbol{x}^2 \neq 0$. In this case, following the Minkowski space example, x is referred to as a space-like vector. Null-like vectors are those with $\|x\| = 0$; that is, $(\boldsymbol{x})^2 = 2x^4x^5$. Therefore, the condition $\|x\| \geq 0$ is physically acceptable, since the movement of the system is in a manifold with the space in both cases given by $(\boldsymbol{x})^2 \geq 0$. For vectors satisfying $\|x\| < 0$, the physically acceptable situations are those for which x^4 and x^5 have the same sign, for $(\boldsymbol{x})^2 < 2x^4x^5$.

Any arbitrary vector in $\mathcal{E}$, say $\boldsymbol{A} = (A^1, A^2, A^3)$, is in correspondence with a vector in $\mathcal{G}$, say A, through the embedding $\Im : \boldsymbol{A} \mapsto A = (\boldsymbol{A}, d, \boldsymbol{A}^2/2d)$, where d is an arbitrary quantity. Indeed, using Eq. (4), it follows that, in this case,

$$\begin{aligned} (A|A) &= \eta_{\mu\nu} A^\mu A^\nu, \\ &= \sum_{i=1}^{3} A^i A^i - 2A^4 A^5 = 0. \end{aligned}$$

That is, according to $\Im$, each vector in $\mathcal{E}$ is in homomorphic correspondence with null-like vectors in $\mathcal{G}$.

$\mathcal{G}$ can still be mapped into a (4, 1)-de Sitter space ($\mathcal{S}$)[9] by the following linear transformation U:[11]

$$U : x^i \mapsto \xi^i = x^i, \;\; i = 1, 2, 3, \tag{8}$$
$$U : x^4 \mapsto \xi^4 = (x^4 + x^5)/\sqrt{2}, \tag{9}$$
$$U : x^5 \mapsto \xi^5 = (x^4 - x^5)/\sqrt{2}. \tag{10}$$

This results in

$$(x|y) = g_{\mu\nu}\xi^\mu\zeta^\nu = (\xi|\zeta), \tag{11}$$

with the (diagonal) metric tensor $(g_{\mu\nu})$ specified by diag$(g_{\mu\nu}) = (+, +, +, -, +)$ (general vectors in $\mathcal{S}$ are denoted by Greek letters as ξ, ζ, ς and so on). In short, we can gather the above results in the following.

Proposition: Using the $\mathcal{G}$ manifold, $\mathcal{E}$ can be embedded into $\mathcal{S}$, a de Sitter space, through the composite mapping $U \circ \Im : \mathcal{E} \mapsto \mathcal{S}$, where the transformation U is given by

$$U = (U^\mu_{\ \nu}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{2}} \\ 0 & 0 & 0 & \dfrac{1}{\sqrt{2}} & \dfrac{-1}{\sqrt{2}} \end{pmatrix}, \tag{12}$$

such that the mapping $U : x^\mu \mapsto \xi^\mu$, $\xi^\mu \in \mathcal{S}$, $x^\mu \in \mathcal{G}$, is given by

$$\xi^\mu = U^\mu_{\ \nu} x^\nu, \tag{13}$$

with $U = U^{-1}$. So, in general, an embedded vector A in $\mathcal{S}$ (from the vector $\boldsymbol{A}$ in $\mathcal{E}$) is given by

$$A = \left(\boldsymbol{A}, \frac{1}{d\sqrt{2}}(2d + \boldsymbol{A}^2), \frac{1}{d\sqrt{2}}(2d - \boldsymbol{A}^2)\right). \qquad \square$$

The transformation matrix U can be used to relate the metric g of $\mathcal{S}$ to η of $\mathcal{G}$. In fact, according to Eq. (11) $(\xi|\zeta) = g_{\mu\nu}\xi^\mu\zeta^\nu$; so we obtain from Eq. (13),

$$(\xi|\zeta) = U^\mu_{\ \rho} g_{\mu\nu} U^\nu_{\ \gamma} x^\rho x^\gamma. \tag{14}$$

Then using Eq. (4), we have

$$\eta_{\rho\gamma} = U^\mu_{\ \rho} g_{\mu\nu} U^\nu_{\ \gamma}, \tag{15}$$

or its inverse, $g = U\eta U$.

It is worth observing that we can define another, more restricted, embedding in

$\mathcal{G}$ space by $\Im' : \boldsymbol{A} \;\mapsto\; A = (\boldsymbol{A}, e, 0)$. Here e is an arbitrary quantity. In this case, on the other hand, A is no longer a null-like vector for $(A|A) = \boldsymbol{A}^2$. In $\mathcal{S}$ space such a vector is written as $A = (\boldsymbol{A}, e/2, e/2)$.

Simple examples of the two kinds of embedded vectors in $\mathcal{G}$ are provided by

$$P = (\boldsymbol{P}, m, E), \tag{16}$$
$$x = (x, t, 0), \tag{17}$$

where, in the former, $E = \boldsymbol{P}^2/2m$ is the energy and $d = m$, while in the latter $e = t$.

Now we will explore linear transformations in the space $\mathcal{G}$. Let

$$\overline{x}^\mu = G^\mu{}_\nu x^\nu \tag{18}$$

be a homogeneous linear transformation such that the metric tensor $\eta_{\mu\nu}$ and the inner product, Eq. (4), are invariant. Then

$$G\eta G^T = \eta, \tag{19}$$

where G^T is the transposed matrix of G.

Consider infinitesimal transformations of the connected part of G, i.e., with $G^\mu{}_\nu = \delta^\mu{}_\nu + \epsilon^\mu{}_\nu$, with $|G| = 1$. Using Eq. (19) we obtain

$$\epsilon^\alpha{}_\nu \eta_{\alpha\beta} + \eta_{\nu\alpha}\epsilon^\alpha{}_\beta = 0. \tag{20}$$

From the analysis of Eq. (20), the matrix $(\epsilon^\mu{}_\nu)$ can be written as

$$(\epsilon^\mu{}_\nu) = \begin{pmatrix} 0 & \epsilon^1{}_2 & \epsilon^1{}_3 & \epsilon^1{}_4 & \epsilon^1{}_5 \\ -\epsilon^1{}_2 & 0 & \epsilon^2{}_3 & \epsilon^2{}_4 & \epsilon^2{}_5 \\ -\epsilon^1{}_3 & -\epsilon^2{}_3 & 0 & \epsilon^3{}_4 & \epsilon^3{}_5 \\ \epsilon^1{}_5 & \epsilon^2{}_5 & \epsilon^3{}_5 & \epsilon^4{}_4 & 0 \\ \epsilon^1{}_4 & \epsilon^2{}_4 & \epsilon^3{}_4 & 0 & -\epsilon^4{}_4 \end{pmatrix}. \tag{21}$$

Defining

$$\begin{aligned} &\epsilon^1{}_2 = m^3,\; \epsilon^1{}_3 = m^2,\; \epsilon^2{}_3 = m^1, \\ &\epsilon^1{}_4 = n^1,\;\; \epsilon^2{}_4 = n^2,\;\; \epsilon^3{}_4 = n^3, \\ &\epsilon^1{}_5 = u^1,\;\; \epsilon^2{}_5 = u^2,\;\; \epsilon^3{}_5 = u^3, \\ &\epsilon^4{}_4 = u, \end{aligned}$$

the matrix $(\epsilon^\mu{}_\nu)$, Eq. (21), can be written as

$$(\epsilon^\mu{}_\nu) = \sum_{i=1}^{3} m^i L_i + \sum_{i=1}^{3} n^i B_i + \sum_{i=1}^{3} u^i C_i + uD, \tag{22}$$

where

$$L_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$L_3 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$B_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \quad B_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix},$$

$$C_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$C_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

The commutation relations among these generators, $L_1, \cdots, D$, give raise to the following algebraic relations:

$$\begin{aligned} &[L_i, L_j] = \varepsilon_{ijk} L_k, \ \ [L_i, C_j] = \varepsilon_{ijk} C_k, \ \ [L_i, B_j] = \varepsilon_{ijk} B_k, \\ &[B_i, C_j] = \varepsilon_{ijk} L_k - D\delta_{ij}, \ \ [B_i, D] = B_i\,, \ [D, C_i] = C_i. \end{aligned} \tag{23}$$

In order to study representations of such a Lie algebra, we can take advantage of the fact that these generators are connected with those of linear transformations in the de Sitter space, $\mathcal{S}$, since the de Sitter coordinates, ξ^μ, are connected to those of the $\mathcal{G}$-space, x^μ, by Eq. (13). Then, a linear transformation in the $\mathcal{G}$-space, characterized by $G^\mu{}_\nu = \delta^\mu{}_\nu + \epsilon^\mu{}_\nu$, induces a transformation $\widetilde{G}^\mu{}_\nu$ in $\mathcal{S}$ specified by

$$\widetilde{G}^\mu{}_\nu = U^\mu{}_\alpha G^\alpha{}_\beta U^\beta{}_\nu,$$

where S is given by Eq. (12). To proceed further, let us write the algebra given by Eq. (23) in a covariant form, i.e.,

$$[M_{\alpha\beta}, M_{\gamma\rho}] = i(\eta_{\alpha\rho} M_{\beta\gamma} - \eta_{\alpha\gamma} M_{\beta\rho} + \eta_{\beta\gamma} M_{\alpha\rho} - \eta_{\beta\rho} M_{\alpha\gamma}), \tag{24}$$

with $\alpha, \beta, \cdots = 1, \cdots, 5$, such that

$$\begin{aligned} iM_{ij} &= \varepsilon_{ijk} L_k, \\ iM_{i4} &= M_{4i} = B_i, \\ iM_{i5} &= M_{5i} = C_i, \\ iM_{45} &= M_{54} = D. \end{aligned}$$

Another representation for these operators is

$$M_{\alpha\beta} = -i \left(x_\alpha \frac{\partial}{\partial x_\beta} - x_\beta \frac{\partial}{\partial x_\alpha} \right).$$

Using the transformation U, the de Sitter Lie algebra and its Casimir invariants are derived. This implies, if we define $\widetilde{M} = UMU$, with $\widetilde{M} \in \mathcal{S}$, that we obtain

$$[\widetilde{M}_{\alpha\beta}, \widetilde{M}_{\gamma\rho}] = i(g_{\alpha\rho}\widetilde{M}_{\beta\gamma} - g_{\alpha\gamma}\widetilde{M}_{\beta\rho} + g_{\beta\gamma}\widetilde{M}_{\alpha\rho} - g_{\beta\rho}\widetilde{M}_{\alpha\gamma}), \tag{25}$$

which has two Casimir invariants,[9)]

$$I_1 = \widetilde{M}_{\alpha\beta}\widetilde{M}^{\alpha\beta},$$

and

$$I_2 = W_\alpha W^\alpha,$$

with

$$W_\alpha = \varepsilon_{\alpha\beta\gamma\sigma\rho}\widetilde{M}^{\beta\gamma}\widetilde{M}^{\sigma\rho}.$$

Here $\varepsilon_{\alpha\beta\gamma\sigma\rho}$ is the totally antisymmetric tensor in five dimensions. Observe that now $\widetilde{M} \in \mathcal{S}$ (not M) is antisymmetric.

As an example, consider a particular case of rotations plus spatial translations in $\mathcal{G}$ of the type $\bar{x}^\mu = G^\mu{}_\nu x^\nu + a^\mu$, with the infinitesimal part of $G^\mu{}_\nu$ being determined by $\boldsymbol{L}$ and $\boldsymbol{B}$, generators of a Lie algebra of the Euclidean group. (We can consider full inhomogeneous transformations in $\mathcal{G}$, but these are not of much interest here. A more detailed discussion about this point will appear elsewhere). In this case we obtain the algebra

$$\begin{aligned} &[L_i, L_j] = \varepsilon_{ijk} L_k, \;\; [L_i, P_j] = \varepsilon_{ijk} P_k, \;\; [L_i, B_j] = \varepsilon_{ijk} B_k, \\ &[B_i, P_4] = P_i, \;\; [B_i, P_j] = P_5 \delta_{ij}. \end{aligned} \tag{26}$$

Finite transformations are provided by

$$\begin{aligned} &K_i = e^{-v^i B_i}, \;\; R_{ij} = e^{\epsilon_{ijk}\theta_{ij} L_k}, \\ &T_\mu = e^{a^\mu P_\mu}, \end{aligned}$$

where no sum is implied by the repeated indices. Consider a vector in $\mathcal{G}$ given by $x = (\boldsymbol{x}, t, \boldsymbol{x}^2/2t)$. Then the components of the transformed vector, $\bar{x}$, are

$$\bar{x}^i = R^i{}_j x^j - v^i x^4 + a^i, \tag{27}$$

$$\bar{x}^4 = x^4 + a^4, \tag{28}$$

$$\bar{x}^5 = x^5 - v^i (R^i{}_j x^j) + \frac{1}{2}\boldsymbol{v}^2 x^4 + a^5. \tag{29}$$

Equations (27) and (28) are just the Galilei transformations, Eqs. (1) and (2), when $x^4 = t,\ a^4 = b$.

At this point, it is interesting to observe that the natural representation of the Galilei Lie algebra is obtained from Eq. (1) and (2) with $P_5 = 0$. However, P_5 is a Casimir invariant (having a constant value in the representation). Then, in this covariant context, the usual central extention of the Galilei Group arises naturally, without any reference to ray or unfaithful representations.

Another example of this formalism is described by the Lagrangian [11)]

$$\mathcal{L} = -\frac{\hbar^2}{2m}\Big\{\nabla\chi^* \cdot \nabla\chi - \partial_5\chi^*\partial_4\chi - \partial_4\chi^*\partial_5\chi \\ +B^*(x)\left(\partial_5 + \frac{im}{\hbar}\right)\chi + \left(\partial_5 + \frac{im}{\hbar}\right)\chi^* B(x)\Big\},$$

where $B(x)$ is an auxiliary field. Following Ref. 11), the scalar Schrödinger equation is derived. We have

$$\partial_\mu\partial^\mu\chi(x) = 0 \tag{30}$$

and

$$\left(\partial_5 + \frac{im}{\hbar}\right)\chi(x) = 0.$$

Then, $\chi(x) = \exp(-imx^5/\hbar)\psi(\boldsymbol{x}, x^4)$. Since $x^4 = t$, from Eq. (30) we obtain

$$i\hbar\partial_t\psi(\boldsymbol{x}, t) = -\frac{\hbar^2}{2m}\nabla^2\psi(\boldsymbol{x}, t).$$

The energy-momentum tensor is thus

$$T^\alpha_\beta(x) = -\frac{\partial\mathcal{L}}{\partial(\partial_\alpha\chi)}\partial_\beta\chi + \mathcal{L}\delta^\alpha_\beta.$$

Then, the dynamical variables for space translation, P^i, time translation, H, mass, M, space rotation, L^i, and Galilei boost, B^i, are given by

$$\begin{aligned} P^i &= \int d^3x\, dx^5 T^4_i, \\ H &= \int d^3x\, dx^5 T^4_4, \\ M &= \int d^3x\, dx^5 T^4_5, \\ L^i &= \frac{1}{2}\varepsilon_{ijk}\int d^3x\, dx^5 (x^j T^4_k - x^k T^4_j), \\ B^i &= \int d^3x\, dx^5 (tT^4_i + x^i T^4_5), \end{aligned}$$

with

$$\begin{aligned} T^4_i &= \frac{i\hbar}{2}(\chi^*\partial_i\chi - \partial_i\chi^*\chi), \\ T^4_4 &= \frac{\hbar^2}{2m}\nabla\chi^* \cdot \nabla\chi, \\ T^4_5 &= m\,\chi^*\chi. \end{aligned}$$

Using the commutation relation between the fields as given in Ref. 11), it is easy to show that the operators P^i, H, M, L^i and B^i define a representation for the algebra given by Eq. (26).

Finally, we would like to point out that the tensor analysis in $\mathcal{G}$ follows in the usual manner.[12] First, consider $w = w_\mu e^\mu$ and $v = v_\mu e^\mu$; $w, v \in \mathcal{G}^*$. The tensor product of two arbitrary vectors x and y in $\mathcal{G}$ is a bilinear form defined by the mapping $\tau = x \otimes y : \mathcal{G}^* \times \mathcal{G}^* \mapsto \mathcal{R}$, with

$$\tau(w, v) = x \otimes y(w, v) = w(x)v(x). \tag{31}$$

In terms of components, Eq. (31) can be written as

$$x \otimes y(w, v) = w_\mu v_\nu x^\mu y^\nu, \tag{32}$$

and the metric $\eta_{\mu\nu}$ can be given by $\eta_{\mu\nu} = e_\mu \otimes e_\nu$. This is the case since we consider the metric as the mapping $\eta_{\mu\nu} : \mathcal{G}^* \times \mathcal{G}^* \mapsto \mathcal{R}$ such that $\eta_{\mu\nu} : (w, v) \mapsto w_\mu v_\nu$. Then, it follows from Eq. (32) that

$$x \otimes y = x^\mu y^\nu e_\mu \otimes e_\nu. \tag{33}$$

Using Eq. (31), we can show that $\tau^{\mu\nu} \equiv \tau(e^\mu, e^\nu) = x^\mu y^\nu$; as a consequence, $\tau = \tau^{\mu\nu}\eta_{\mu\nu}$. The set $\{\eta_{\mu\nu} = e_\mu \otimes e_\nu\}$ is a basis spanning the vector space defined by the set of 2nd order contravariant tensors; the proof and generalization to higher order tensors are straightforward.

In summary, through an immersion of the Euclidian space in a (4, 1)-de Sitter space, we show how to derive a manifold that leads to a covariant structure of the Galilei symmetries. For instance, for the Euclidian space of positions, time can be identified as an embedding parameter, or, in other words, the classical space-time, $\mathcal{R}^3 \times \mathcal{R}$, is embedded in (4, 1)-de Sitter space. This realizes the natural representation of the Galilei group within the defining representation of the de Sitter Group. We have studied, therefore, a covariant Galilei Lie algebra and developed the manifold analysis. As an example, the structure of the scalar field is considered, resulting in the scalar Schrödinger equation. A more detailed discussion of the connection established here between the Galilei symmetries and the de Sitter geometric spaces is in preparation.

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